THE SPECIAL THEORY OF RELATIVITY -

A CLASSICAL APPROACH

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ABSTRACT

The purpose of this paper is to present a new simplified approach to the mathematical formulation of Albert Einstein's Special Theory of Relativity. Initially, a new representation of Hermann Minkowski's Pseudo-Euclidean Space-Time "World" is defined, within which a new concept, Existence Velocity, is incorporated. This then enables the simplified development, along classical analytical lines, of the Special Theory's relativistic kinematic and kinetic relationships.

MATHEMATICAL NOMENCLATURE

Mathematical presentation is as follows:

(i) All mathematical characters, are in Italic Times New Roman font. i.e., *x.*

(ii) Axis designators are in capitalised non-italic **bold,** i.e., $\mathbf{X_0}$.
Unit Vectors are in ***italic Bold,*** i.e. $\boldsymbol{n}$

(iii) All separate spatial and temporal vector characters, are in ***italic bold,*** i.e., $\boldsymbol{v}$, except where a Greek character is used where it is then over barred.

(iv) All combined spatial-temporal vector characters are in non-italic bold characters of a different font, (TimpaniHeavy), i.e **S.**

(v) Differentials are represented by either the usual, *d(func)/d(var)* or sometimes for clarity by the dot form, $\dot{r}$, for the first differential and, $\ddot{r}$ for the second. The latter is however, only used for differentials with respect to a time variable, where the function differentiated is not itself a function of time.

CONTENTS

The Special Theory of Relativity - A Classical Approach

1. Introduction

In 1905 Albert Einstein published his paper on the Special Theory of Relativity, which, as is well known, is concerned with the characteristics of space, time and matter when a mass possesses a constant velocity in Pseudo - Euclidean Space - Time. Subsequently, Hermann Minkowski showed that Pseudo - Euclidean Space - Time could be represented by a four dimensional `World', in which three dimensions were spatial in nature, and the fourth, temporal. A point within such a `World' was said to exist at the co-ordinate positions representing its location. Minkowski's development subsequently led to the mathematical formulation of the Special Theory using such tools as the Tensor Calculus.

By utilising Minkowski's representation of space-time in a new way e.g. as a linear complex spatial/temporal manifold, in which the temporal dimension is represented as the imaginary part and the spatial dimensions as the real part, and the introduction within it of a new concept, Existence Velocity, a Space - Time Domain designated $\mathbf{D_0}$ is created. The simple process of induced spatial rectilinear motion within this Domain then permits the derivation, using classical analytical methods, of the main kinematic and kinetic relationships extant within the Special Theory, together with a number of new ones. As a demonstration of applicability, the concept is then extended to planar, and central orbital motions. Finally, via conformance to the appropriate criteria, the Domain $\mathbf{D_0}$ is subsequently shown in Appendix A, to be equivalent to Pseudo - Euclidean Space - Time. This is augmented by the reduction of selected derivations to their classical equivalents in Appendix B.

2. The Space-Time Domain $\mathbf{D_0}$

2.1 Definition

The Domain $\mathbf{D_0}$ is defined as a space time of four mutually orthogonal linear dimensions, three of which, $\mathbf{X_1}$, $\mathbf{X_2}$ and $\mathbf{X_3}$ are spatial, while the fourth, $\mathbf{X_o}$, following Minkowski, is temporal. $\mathbf{X_o}$ is defined, and will be shown, (Appendix A), to be the product of the time t in $\mathbf{D_0}$ and a constant velocity parameter c, designated the Spatial Terminal Velocity of $\mathbf{D_0}$.
The Domain $\mathbf{D_0}$ is further characterised in that for any spatial - temporal point to exist within it, that point must at all times possess a vector velocity, designated Existence Velocity, the magnitude of which has the same value as the Spatial Terminal Velocity. Thus, for any point to exist within $\mathbf{D_0}$, it is necessary for the magnitude of the vectorial sum of its velocities along the four co-ordinate axes, to be at all times, equal to c.

2.2 Existence within $\mathbf{D_0}$

The position of any random point B within $\mathbf{D_0}$ relative to some chosen reference can be expressed in spatial - temporal vector form as

$$\mathbf{S} = \boldsymbol{i}\,x_1 + \boldsymbol{l}\,x_2 + \boldsymbol{k}\,x_3 + \boldsymbol{j}\,x_o \qquad (2.1)$$

where x_1, x_2 and x_3 are each a distance along the corresponding spatial axes of $\mathbf{D_0}$, for which $\boldsymbol{i}$, $\boldsymbol{l}$ and $\boldsymbol{k}$ are the appropriate unit vectors. x_o is a distance along the temporal axis for which $\boldsymbol{j}$ is the unit vector. $\boldsymbol{i}$, $\boldsymbol{l}$ and $\boldsymbol{k}$ each have the usual magnitude of unity, while $\boldsymbol{j}$ has the magnitude of $\sqrt{-1}$.
As the temporal axis of $\mathbf{D_0}$ is the product of the constant c with the time t, so then will the distance x_o be a product of c and some function of the time t in $\mathbf{D_0}$. Accordingly (2.1) may be rewritten as

$$\mathbf{S} = \boldsymbol{r} + \boldsymbol{j}\,ct_p \qquad (2.2)$$

where t_p is a function of the time t in $\mathbf{D_0}$ and, also, where the spatial component of (2.1) has been replaced with its resultant spatial vector position on a polar spatial linear co-ordinate axis $\mathbf{R}$.
The velocity of such a point in $\mathbf{D_0}$ is defined by differentiating (2.2) with respect to t thus:

$$\mathbf{V} = \boldsymbol{v} + \boldsymbol{j}\,c\frac{dt_p}{dt} \qquad (2.3)$$

where $\mathbf{V} = d\mathbf{S}/dt$ and $\boldsymbol{v} = d\boldsymbol{r}/dt$

Invoking the characteristic of existence in $\mathbf{D_0}$, (2.3) must at all times conform to the following identity,

$$V = c \qquad (2.4)$$

where V is the magnitude of $\mathbf{V}$. Substitution of (2.4) into (2.3) gives, after taking the magnitude,

$$c^2 = v^2 + c^2\left(\frac{dt_p}{dt}\right)^2 \qquad (2.5)$$

so that

$$\frac{dt_p}{dt} = \left(1 - \frac{v^2}{c^2}\right)^{1/2} \tag{2.6}$$

where v is the magnitude of $\boldsymbol{v}$. Substitution of (2.6) into (2.3) then gives

$$\mathbf{V} = \boldsymbol{v} + \boldsymbol{j}\,c\left(1 - \frac{v^2}{c^2}\right)^{1/2} \tag{2.7}$$

and the following terms are defined thus:

- $\mathbf{V}$ is the Existence Velocity of the the point B in $\mathbf{D_0}$
- dt_p/dt is the Temporal Rate of the the point B in $\mathbf{D_0}$
- $c\,dt_p/dt$ is the Temporal Velocity of the the point B in $\mathbf{D_0}$ and,
- t_p is the Proper Time of the point B in $\mathbf{D_0}$. Thus t_p is the time measured by any observer moving with a spatial velocity v in $\mathbf{D_0}$.

From (2.7) it is evident that $\mathbf{V}$ possesses a spatial - temporal orientation in $\mathbf{D_0}$ which is directly dependent upon the spatial velocity magnitude v. As v increases from zero, temporal velocity undergoes a proportional reduction such that $\mathbf{V}$, relative to the temporal co-ordinate of $\mathbf{D_0}$ rotates through an angle in the $\mathbf{X_0}$ - $\mathbf{R}$ plane, related to v by the expression

$$Sin\,\theta = \frac{v}{c} \tag{2.8}$$

Thus, for future reference it is noted that

$$Cos\,\theta = \left(1 - \frac{v^2}{c^2}\right)^{1/2} = \frac{dt_p}{dt} \tag{2.9}$$

This completes the definition and characterisation of $\mathbf{D_0}$. Its equivalence with Pseudo-Euclidean Space-Time is demonstrated in Appendix A. The next section formulates the kinematics and kinetics of rectilinear motion within $\mathbf{D_0}$ for comparison with that of the Special Theory.

3. The Mechanics of Simple Spatial Rectilinear Motion in $\mathbf{D_0}$

3.1 Mass and The Equation of Motion

The Special Theory of Relativity asserts that the mass of a fixed quantity of matter, spatially in motion with a constant rectilinear velocity in Pseudo - Euclidean Space - Time, is greater than when it is at rest. For this to be so the increase in mass can only take place during the time that spatial acceleration is in effect. This process can be investigated by treating mass as a variable when analysing the change in the Existence Momentum of a point mass subjected to spatial acceleration in $\mathbf{D_0}$.

If m is the mass of the point mass with Existence Velocity $\mathbf{V}$ in $\mathbf{D_0}$, then its Existence Momentum will be given by:

$$\mathbf{M} = m\mathbf{V} = m\left[\boldsymbol{v} + \boldsymbol{j}\,c\left(1-\frac{v^2}{c^2}\right)^{1/2}\right] \tag{3.1}$$

where $\mathbf{V}$ has been substituted from (2.7). If $\mathbf{F}$ is the force applied to effect acceleration then:

$$\mathbf{F} = \frac{d\mathbf{M}}{dt} = m\dot{\boldsymbol{v}} + \boldsymbol{v}\dot{m} + \boldsymbol{j}\left[\dot{m}c\left(1-\frac{v^2}{c^2}\right)^{1/2} - \frac{mv\dot{v}}{c\left(1-\frac{v^2}{c^2}\right)^{1/2}}\right] \tag{3.2}$$

Showing that there are four kinetic reaction terms involved in this process.

If $\mathbf{F}$ is purely spatial, the temporal component of (3.2) will be zero, whereby:

$$\dot{m}c\left(1-\frac{v^2}{c^2}\right)^{1/2} = \frac{mv\dot{v}}{\left(1-\frac{v^2}{c^2}\right)^{1/2}} \tag{3.3}$$

so that upon separating variables and integrating

$$\boldsymbol{ln}\,m = -\boldsymbol{ln}\left(1-\frac{v^2}{c^2}\right)^{1/2} + k \tag{3.4}$$

The constant of integration is obtained from initial conditions viz: when $v = 0$, $m = m_o$, the mass of the point mass when spatially at rest in $\mathbf{D_0}$, i.e. the 'rest mass'. Then:

$$k = \boldsymbol{ln}\,m_0 \tag{3.5}$$

which gives in (3.4)

$$m = \frac{m_0}{\left(1-\frac{v^2}{c^2}\right)^{1/2}} \tag{3.6}$$

as asserted in the Special Theory. However, it is clear from the above development that, in addition to a constant spatial velocity, (3.6) is also valid during the time that spatial acceleration of a point mass is in effect. For reasons that will be discussed later *m* will be referred to as Energy Mass.

Substitution of (3.6) into (3.3) yields:

$$\dot{m} = \frac{m_0 v\dot{v}}{c^2\left(1-\frac{v^2}{c^2}\right)^{3/2}} \tag{3.7}$$

which thus represents the time rate of change of mass subjected to spatial acceleration in $\mathbf{D_0}$. This can also be obtained by simply differentiating (3.6) with respect to the time *t*. These last two terms, (3.6) and (3.7), may now be inserted into (3.2), whereupon the temporal component vanishes and, if rectilinear motion only is being considered, $\mathbf{F}$ can also be reduced to a spatial vector, $\boldsymbol{F}$, so that (3.2) yields, after reduction:

$$\boldsymbol{F} = \frac{\dot{\boldsymbol{v}}\, m_0}{\left(1-\frac{v^2}{c^2}\right)^{3/2}} \tag{3.8}$$

As $\boldsymbol{F}$ is arbitrary, (3.8) represents the spatial rectilinear equation of motion of a point mass in $\mathbf{D_0}$. (Non rectilinear motion is examined in Sections 4 and 5).

Note that the right hand side of (3.8) is the product of the spatial acceleration and a mass term.

Putting

$$m_a = \frac{m_0}{\left(1-\frac{v^2}{c^2}\right)^{3/2}} \tag{3.9}$$

then m_a is, from (3.8), synonymous with inertial mass. Three values of mass have thus been identified for the same point mass i.e.

m_o	- Rest Mass
$m = \frac{m_0}{\left(1-\frac{v^2}{c^2}\right)^{1/2}}$	- Energy Mass
$m_a = \frac{m_0}{\left(1-\frac{v^2}{c^2}\right)^{3/2}}$	- Inertial Mass

The latter two are sometimes referred to in the literature [1], [2], [4] as 'transverse' and 'longitudinal' mass, (see Section 4).

For interpretation of these results, reference is made to Fig. 3.1. where it is shown that as a consequence of the rotation of $\mathbf{V}$ in $\mathbf{D}_\sigma$ the applied spatial vector force $\boldsymbol{F}$ may be resolved from two spatial - temporal vector components, $\mathbf{F_a}$ normal to $\mathbf{V}$ and $\mathbf{F_e}$ parallel to $\mathbf{V}$.

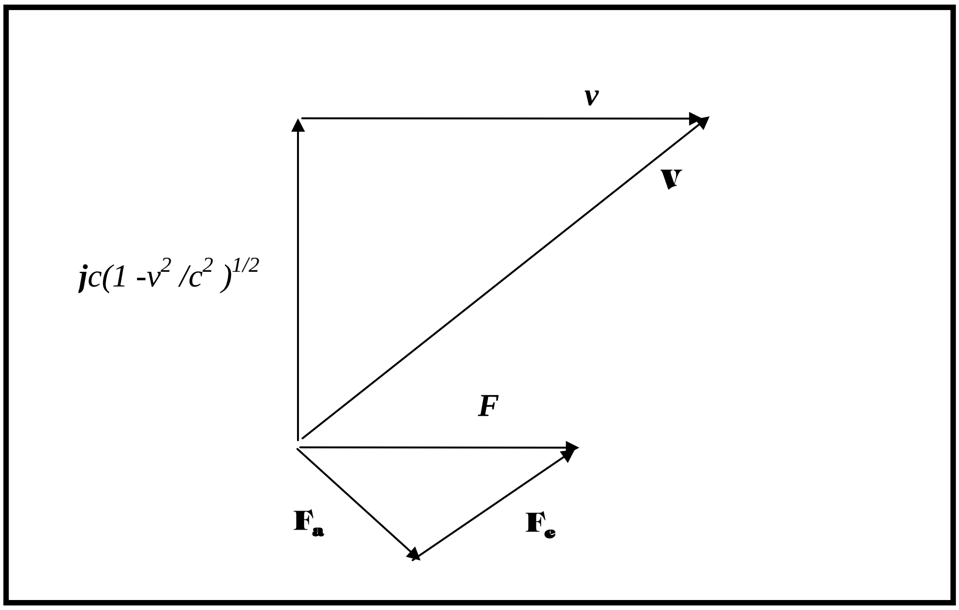

FIG. 3.1: COMPONENTS OF F WITH RESPECT TO $\mathbf{V}$

The component $\mathbf{F_a}$ is proportional to the change in Existence Velocity, while the component $\mathbf{F_e}$ is proportional to the change in the mass. This is clear from (3.2) where by inspection:

$$\mathbf{F_a} = m\frac{d\boldsymbol{v}}{dt} - \mathbf{j}\left(\frac{mv\dfrac{dv}{dt}}{c\left(1-\dfrac{v^2}{c^2}\right)^{1/2}}\right) \tag{3.10}$$

$$= m\frac{d\mathbf{V}}{dt} \tag{3.11}$$

and

$$\mathbf{F_e} = \boldsymbol{v}\frac{dm}{dt} + \boldsymbol{j}c\left(1-\frac{v^2}{c^2}\right)^{1/2}.\frac{dm}{dt} \tag{3.12}$$

$$= \mathbf{V}\frac{dm}{dt} \tag{3.13}$$

From these equations, (3.2) can now be interpreted kinetically. From (3.10) and (3.12) it can be seen that the kinetic reactions to the two components of $\boldsymbol{F}$, each comprise a spatial and temporal term. If the balanced force vector $\boldsymbol{F}$ is diagrammatically represented as in Fig. 3.2, the four kinetic reaction terms of (3.2) can be interpreted as follows:

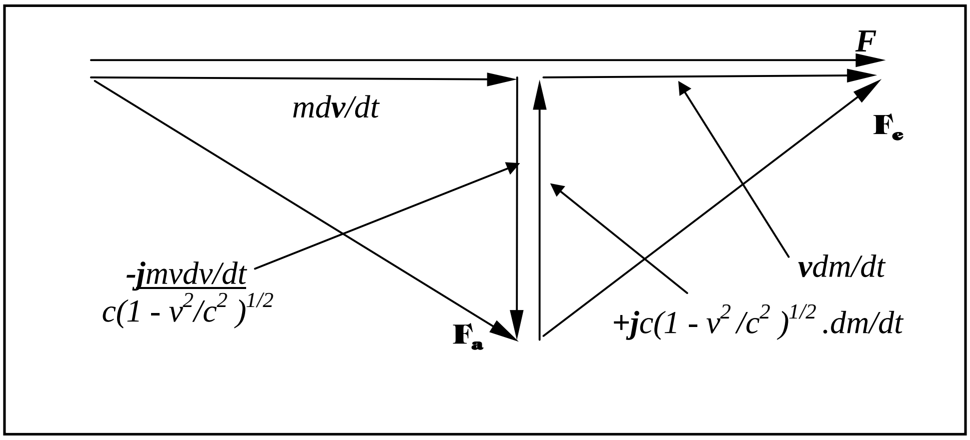

FIG 3.2: THE BALANCED FORCE VECTOR

(i) The spatial term, $m\,d\mathbf{v}/dt$ is the reaction force of the energy mass to spatial acceleration.

(ii) The temporal term, $\dfrac{-\boldsymbol{j}\,mv\,dv/dt}{c\left(1-v^2/c^2\right)^{1/2}}$ is the reaction force of the energy mass to temporal deceleration.

(iii) The temporal term, $\boldsymbol{j}\,c\left(1-v^2/c^2\right)^{1/2}.dm/dt$ is a reaction force generated by the combination of mass rate and temporal velocity and acts in opposition to the term in (ii).

(iv) The spatial term, $\mathbf{v}\,dm/dt$ is a reaction force generated by the combination of mass rate and spatial velocity and acts as an additional reaction to spatial acceleration, thereby causing the apparent mass to increase, from m to m_a, during the period of acceleration. A consequence of this, is that this last term must be related to the difference between inertial and energy mass, by the product of that difference and the spatial acceleration. This may be shown as follows.

From (3.6) and (3.9).

$$\begin{aligned} m_a - m &= \frac{m_0}{\left(1-\dfrac{v^2}{c^2}\right)^{3/2}} - \frac{m_0}{\left(1-\dfrac{v^2}{c^2}\right)^{1/2}} \\ &= \frac{m_0 v^2}{c^2\left(1-\dfrac{v^2}{c^2}\right)^{3/2}} \end{aligned} \tag{3.14}$$

which with insertion of (3.7) gives the required relationship:

$$\left(m_a - m\right)\frac{dv}{dt} = v\,dm/dt \tag{3.15}$$

Finally, a comment upon two additional points that emerge from the above analysis. Firstly the fact that the two temporal reaction terms, items (ii) and (iii) above, are, as shown by (3.2) and in Fig. (3.2), to be equal in magnitude but opposite in sign, does not mean that they do not separately exist. Whilst they do in the above example, cancel, they are quite different in nature, the first being a mass reaction to temporal deceleration and the second a mass rate reaction to temporal velocity. They are equal in magnitude but opposite in sign solely because, in this case, there is no net temporal component of impressed force, i.e. $\mathbf{F}$ is purely spatial.

The second point concerns the inertial mass m_a. While m_a can be expressed solely as a function of the spatial velocity, as is apparent from (3.9), it is equally apparent from (3.15) that its existence is entirely dependent upon the spatial reaction term vdm/dt. As this term only exists while spatial acceleration is taking place, so then can m_a only exist during this period. When spatial acceleration ceases, the term vdm/dt vanishes and, the value of the mass instantly reverts to that of the energy mass, m.

3.2 Energy

In classical mechanics the acceleration of a point mass is said to result in it gaining a kinetic energy equal to the product of the applied force and the distance over which it acts. The manner in which kinetic energy was stored by such an accelerated mass was not addressed. From the results of the preceding section however, this can now be demonstrated as follows.

Consider the change in energy of the point mass as the spatial acceleration proceeds. Integrating (3.8) with respect to the spatial distance travelled,

$$E_k = \int F dr$$

$$= \int m_a \frac{dv}{dt} dr \qquad (3.16)$$

which from (3.9) becomes

$$E_k = m_0 \int \frac{v dv}{\left(1 - \frac{v^2}{c^2}\right)^{3/2}} \qquad (3.17)$$

using simple substitution methods this evaluates to

$$E_k = \frac{m_0 c^2}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}} + k \qquad (3.18)$$

Initial conditions are that $E_k = 0$ when $v = 0$ so that $k = - m_0 c^2$. Inserting this into (3.18) then gives a version of Einstein's well known equation for the energy of matter.

$$E_k = mc^2 - m_0 c^2 \qquad (3.19)$$

where each term may be interpreted, as in the literature, as follows:

(i) mc^2 is the total energy of matter at some instantaneous spatial velocity v.

(ii) E_k is the energy imparted to matter via the action of the applied force over the spatial distance travelled during its application, i.e. kinetic energy.

(iii) $m_0 c^2$ is the rest mass energy of matter

From (3.19) it is seen that the increase in mass, from m_0, that at rest, to m, that at velocity v, is as described in the literature, [2], due to the storage of energy imparted from the applied force. Thus m is the mass equivalent of the total energy of matter at the instantaneous velocity v. It was for this reason that m was earlier designated as Energy Mass.

Reduction of all of the above relationships involving the spatial velocity, v, to non-relativistic form is effected in the usual manner, by assuming the Spatial Terminal Velocity, c, to be infinitely large, (see Appendix B).

4 Curvi-Linear Motion in $\mathbf{D_0}$ - Primary Equations

This condition is briefly investigated to illustrate the effects of accelerative forces on the direction of motion of a point mass in $\mathbf{D_0}$. In doing so however, it also enables the reason for the original designations of "longitudinal" and "transverse" mass to be simply demonstrated (see Sect.3.1). The spatial situation can most easily be described by Fig.4.1. where, for clarity, spatial cartesian co-ordinates are now represented by $\mathbf{X}$ and $\mathbf{Y}$.

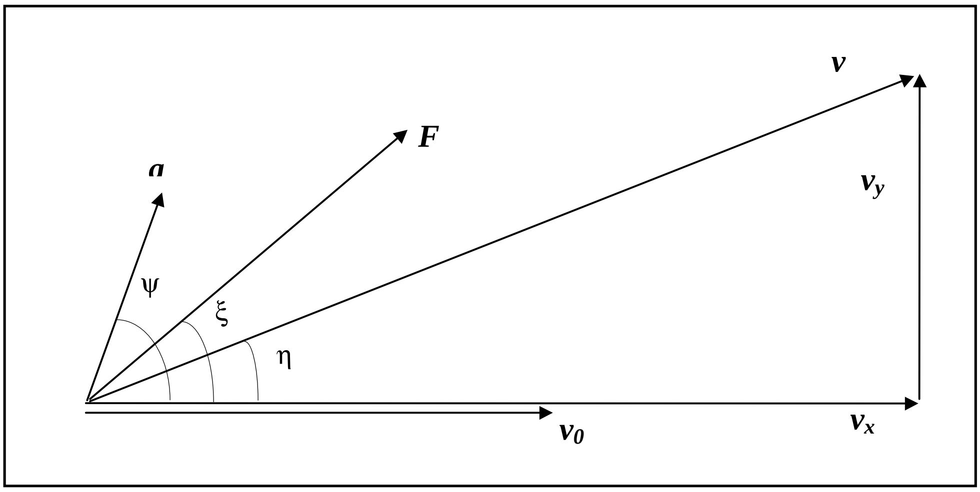

Fig. 4.1 - Force/Velocity/Acceleration Diagram (Spatial)
Non-Rectilinear Acceleration in $\mathbf{D_0}$

The momentum equations are:

$$M_x = mv_x \tag{4.1}$$

$$M_y = mv_y \tag{4.2}$$

$$M_t = mc\left(1-\frac{v^2}{c^2}\right)^{1/2} \tag{4.3}$$

where m is the energy mass of the point mass and M_x and M_y are the respective spatial, and M_t the temporal, existence momentums. The initial velocity at $t = 0$ is v_0, the spatial acceleration is a, and the other terms are as shown in Fig.4.1. Differentiating with respect to t gives the force equations thus:

$$F_x = v_x \frac{dm}{dt} + m\frac{dv_x}{dt} \tag{4.4}$$

$$F_y = v_y \frac{dm}{dt} + m\frac{dv_y}{dt} \tag{4.5}$$

$$F_t = c\left(1-\frac{v^2}{c^2}\right)^{1/2} \cdot \frac{dm}{dt} - \frac{mv\frac{dv}{dt}}{c\left(1-\frac{v^2}{c^2}\right)^{1/2}} \tag{4.6}$$

Determination of *m* and *dm/dt*

If in (4.6), F_t is zero, *m* can be determined by simple integration to give, as in Sect. 3:

$$m = \frac{m_0}{\left(1-\frac{v^2}{c^2}\right)^{1/2}} \tag{4.7}$$

so that also

$$\frac{dm}{dt} = \frac{m_0 v\frac{dv}{dt}}{c^2\left(1-\frac{v^2}{c^2}\right)^{3/2}} \tag{4.8}$$

but in this case with

$$v = \left(v_x^2 + v_y^2\right)^{1/2} \tag{4.9}$$

then

$$\frac{dv}{dt} = \frac{\left(v_x\frac{dv_x}{dt} + v_y\frac{dv_y}{dt}\right)}{v} \tag{4.10}$$

and this in (4.8) gives:

$$\frac{dm}{dt} = \frac{m_0\left(v_x\frac{dv_x}{dt} + v_y\frac{dv_y}{dt}\right)}{c^2\left(1-\frac{v^2}{c^2}\right)^{3/2}} \tag{4.11}$$

Substitution from (4.19) and (4.20) for dv_x/dt and dv_y/dt then gives:

$$\frac{dm}{dt} = \frac{F}{c^2}\left(v_x\,\boldsymbol{cos}\,\xi + v_y\,\boldsymbol{sin}\,\xi\right) \qquad (4.12)$$

Finally substitution for v_x and v_y from relationships implicit in Fig. 4.1, yields:

$$\frac{dm}{dt} = \frac{F}{c^2}v\left[\boldsymbol{cos}(\xi - \eta)\right] \qquad (4.13)$$

which clearly shows that a variation in mass only results from that element of applied accelerative force acting along the velocity vector.

Determination of dv_x/dt and dv_y/dt and Associated Inertial Masses.

Substitution of (4.7) and (4.11) into (4.4) and (4.5) yields after reduction:

$$F_x = \frac{m_0}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}\left[\left(1-\frac{v_y^2}{c^2}\right)\left(\frac{dv_x}{dt}\right) + \frac{v_x v_y}{c}\left(\frac{dv_y}{dt}\right)\right] \qquad (4.14)$$

$$F_y = \frac{m_0}{\left(1-\frac{v^2}{c^2}\right)^{\frac{3}{2}}}\left[\left(1-\frac{v_x^2}{c^2}\right)\left(\frac{dv_y}{dt}\right) + \frac{v_x v_y}{c}\left(\frac{dv_x}{dt}\right)\right] \qquad (4.15)$$

so that from the F_y half of this equation:

$$\frac{dv_y}{dt} = \frac{1}{\left(1-\frac{v_x^2}{c}\right)}\left[\frac{F_y}{m_0}\left(1-\frac{v^2}{c^2}\right)^{3/2} - \frac{v_x v_y}{c^2}\left(\frac{dv_x}{dt}\right)\right] \qquad (4.16)$$

With substitution of this into F_x, (4.15), there is after reduction:

$$F_x = \frac{m_0\left(\frac{dv_x}{dt}\right)}{\left(1-\frac{v_x^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}} + \frac{F_y v_x v_y}{c^2\left(1-\frac{v_x^2}{c^2}\right)} \qquad (4.17)$$

but from Fig. 4.1, $F_y = F_x\,tan\xi$ which when substituted into (4.17) yields:

$$\frac{dv_x}{dt} = \frac{F}{m_0}\left(1-\frac{v^2}{c^2}\right)^{1/2}\left[\left(1-\frac{v_x^2}{c^2}\right)\boldsymbol{cos}\,\xi - \frac{v_x v_y}{c^2}\boldsymbol{sin}\,\xi\right] \qquad (4.18)$$

Finally substitution for v_x and v_y from relationships implicit in Fig. 4.1, gives:

$$\frac{dv_x}{dt}=\frac{F}{m_0}\left(1-\frac{v^2}{c^2}\right)^{1/2}\left(\boldsymbol{cos}\,\xi-\frac{v^2}{c^2}\boldsymbol{cos}(\xi-\eta)\boldsymbol{cos}\,\eta\right) \tag{4.19}$$

$$\frac{dv_y}{dt}=\frac{F}{m_0}\left(1-\frac{v^2}{c^2}\right)^{1/2}\left(\boldsymbol{sin}\,\xi-\frac{v^2}{c^2}\boldsymbol{cos}(\xi-\eta)\boldsymbol{sin}\,\eta\right) \tag{4.20}$$

Interpretation of these equations is quite simple when (4.7) and (4.13) are introduced. In both cases (4.19) and (4.20) reduce to (4.4) and (4.5), showing that the first term inside the respective brackets is the normal acceleration resulting from the ratio of force to energy mass, while the second term is the retardation due to the reaction between mass rate and spatial velocity. Consequently from (4.19) and (4.20), m_{ax} and m_{ay}, the inertial masses in the respective directions, may be expressed thus:

$$m_{ax}=\frac{F_x}{dv_x/dt}=\frac{m_0}{\left(1-\frac{v^2}{c^2}\right)^{1/2}\left[1-\frac{v^2}{c^2}\boldsymbol{cos}\,\eta\,\boldsymbol{cos}(\xi-\eta)\boldsymbol{sec}\,\xi\right]} \tag{4.21}$$

$$m_{ay}=\frac{F_y}{dv_y/dt}=\frac{m_0}{\left(1-\frac{v^2}{c^2}\right)^{1/2}\left[1-\frac{v^2}{c^2}\boldsymbol{sin}\,\eta\,\boldsymbol{cos}(\xi-\eta)\boldsymbol{cos}\,ec\,\xi\right]} \tag{4.22}$$

The difference between these two terms is solely due to the different mass rate reaction forces generated in the respective directions. Equality occurs when the applied force and velocity vectors are coincident i.e. when ξ and η are equal (but not zero). Then:

$$m_{ax}=m_{ay}=\frac{m_0}{\left(1-\frac{v^2}{c^2}\right)^{3/2}} \tag{4.23}$$

and is of course equal to the inertial mass of rectilinear motion.

Angular Relationship Between Applied Force and Acceleration.

This can most easily be inspected by comparing the respective angular relationships between both the $\boldsymbol{a}$ and $\boldsymbol{F}$ directions and the $\mathbf{X}$ axis. Thus from (4.19) and (4.20)directly:

$$\boldsymbol{tan}\,\psi=\frac{dv_y/dt}{dv_x/dt}=\frac{\boldsymbol{sin}\,\xi-\frac{v^2}{c^2}\boldsymbol{sin}\,\eta\,\boldsymbol{cos}(\xi-\eta)}{\boldsymbol{cos}\,\xi-\frac{v^2}{c^2}\boldsymbol{cos}\,\eta\,\boldsymbol{cos}(\xi-\eta)} \tag{4.24}$$

which shows that the resultant spatial acceleration does not lie in the same direction as the applied force. The reason is again the difference in the mass rate reaction terms, in the $\mathbf{X}$ and $\mathbf{Y}$ directions.

Determination of dv/dt and dv_n/dt

These terms are defined as the accelerations produced both along and normal to the velocity vector. Substitution of (4.19) and (4.20) into (4.10) yields:

$$\frac{dv}{dt}=\frac{F}{m_0}\left(1-\frac{v^2}{c^2}\right)^{3/2}\boldsymbol{cos}(\xi-\eta) \tag{4.25}$$

Note that inertial mass along the velocity vector, is, as would be expected by virtue of (4.13), equal to that in rectilinear acceleration.

To determine dv_n/dt, note that from Fig. 4.1

$$\frac{dv_n}{dt}=\frac{dv}{dt}\boldsymbol{tan}(\psi-\eta) \tag{4.26}$$

after expansion, substitution from (4.24) produces, after some reduction:

$$\frac{dv_n}{dt}=\frac{\frac{dv}{dt}\boldsymbol{tan}(\xi-\eta)}{1-\frac{v^2}{c^2}} \tag{4.27}$$

so that substitution for dv/dt from (4.25) then yields:

$$\frac{dv_n}{dt}=\frac{F}{m_0}\left(1-\frac{v^2}{c^2}\right)^{1/2}\boldsymbol{sin}(\xi-\eta) \tag{4.28}$$

The point about this result is of course the appearance of the energy mass, there being no mass rate reaction involved because the direction concerned is normal to the velocity vector.

Other terms, such as acceleration along and normal to the direction of applied force, angular rate and angular acceleration of the velocity vector and, associated energies, exhibit relationships of a similar nature to the above.

Boundary Conditions of ξ.

The two boundary conditions of ξ are, $\xi = 0$ and $\xi = \pi/2$, at which the following apply:

(i) $\xi = 0$ e.g. $\boldsymbol{F}$ lies parallel to the Velocity Vector, along the $\mathbf{X}$ axis

Motion in the $\mathbf{Y}$ direction is non-existent while that in the $\mathbf{X}$ direction is of course that of rectilinear motion of the main text, i.e. η is also zero. (Note that this condition is a special case of $\xi = \eta$ used to obtain (4.23) above). Of particular interest however is from (4.21) and (4.22)

$$m_{ax} = \frac{m_0}{\left(1 - \frac{v^2}{c^2}\right)^{3/2}} \tag{4.29}$$

$$m_{ay} = \frac{m_0}{\left(1 - \frac{v^2}{c^2}\right)^{1/2}} \tag{4.30}$$

m_{ax} is now the inertial mass of rectilinear motion and applies only in the $\mathbf{X}$ direction, i.e. in the direction of the now coincident force, acceleration and velocity vectors. It was for this reason that m_{ax}, in the above form was in the literature originally termed "longitudinal" mass. Under this condition m_{ay} is the mass applicable normal to the accelerated motion and was consequently originally termed "transverse" mass. It is equal to the value of energy mass because of course no mass rate reaction term exists in the transverse direction.

(ii) $\xi = \pi/2$ e.g. $\boldsymbol{F}$ lies normal to the Initial Velocity Vector.
The most interesting result from this situation emerges from (4.14) and (4.15) i.e.

$$\frac{dv_x}{dt} = -\frac{Fv^2}{m_0 c^2}\left(1 - \frac{v^2}{c^2}\right)^{1/2} \boldsymbol{sin}\,\eta\, \boldsymbol{cos}\,\eta \tag{4.31}$$

$$m_{ax} = 0 \tag{4.32}$$

This simply states that with all the accelerative force applied normal to the direction of initial velocity, a very small deceleration in that direction occurs due to the reaction of mass rate with v_x. Inertial mass in the $\mathbf{X}$ direction consequently takes the `hypothetical' value of zero as a deceleration occurs without the application of an external force in that direction.

Summary

It is clear that the primary difference between the motion described herein and that of Classical Mechanics, apart from the main relativistic effects, is due to the mass rate reaction terms. Most particular in this respect is the non-coincidence of the force and acceleration vectors. In the next Section it is shown that a primary result of this effect upon a trajectory is to cause it to rotate.

5 Planar Orbital Motion Kinematics In $\mathbb{D}_0$

In this Section, the equation of an orbital motion is first derived and then solved as a second illustration of the manner in which the concept of Existence Velocity, may be applied to relativistic problems of this nature in Pseudo - Euclidean Space - Time.

Derivation of the General Curvi - Linear Equation of Motion in a Plane in $\mathbb{D}_0$

Repeating (2.7) for convenience:

$$\mathbb{V} = \boldsymbol{v} + \boldsymbol{j}\,c\left(1-\frac{v^2}{c^2}\right)^{1/2}$$

If the energy mass is *m* then existence momentum, for purely planar motion, is:

$$\mathbb{M} = m\left[\dot{r}\,\boldsymbol{n} + \omega r\,\boldsymbol{t} + \boldsymbol{j}\,c\left(1-\frac{\dot{r}^2}{c^2}-\frac{\omega^2 r^2}{c^2}\right)^{1/2}\right] \tag{5.1}$$

where for mathematical convenience spatial polar axes have been chosen and where, with reference to some stationary origin,

r is the radial distance of the orbit.
$\dot{r} = dr/dt$ is the radial velocity of the orbit.
$\omega = d\varphi/dt$ is the angular rate of the orbit.
$\boldsymbol{n}$ and $\boldsymbol{t}$ are the radial and radial normal unit vectors.

Differentiating (5.1) with respect to time yields the force equation thus:

$$\mathbb{F} = \frac{d\mathbb{M}}{dt} = \left[m\left(\ddot{r} - \omega^2 r\right) + \dot{m}\dot{r}\right]\boldsymbol{n} + \left[\,m\left(2\omega\dot{r} + \dot{\omega}r\right) + \dot{m}\omega r\right]\boldsymbol{t}$$
$$+\,\boldsymbol{j}\left[\dot{m}c\left(1-\frac{\dot{r}^2}{c^2}-\frac{\omega^2 r^2}{c^2}\right)^{1/2} - \frac{m\left(\dot{r}\ddot{r} + \omega\dot{\omega}r^2 + \omega^2 r\dot{r}\right)}{c\left(1-\frac{\dot{r}^2}{c^2}-\frac{\omega^2 r^2}{c^2}\right)^{1/2}}\right] \tag{5.2}$$

If $\mathbb{F}$ is purely spatial, then the temporal part of (5.2) is zero and *m* can be determined by simple integration to be:

$$m = \frac{m_0}{\left(1-\frac{\dot{r}^2}{c^2}-\frac{\omega^2 r^2}{c^2}\right)^{1/2}} \tag{5.3}$$

where m_o is the rest mass. Thus the mass rate is:

$$\dot{m} = \frac{m_o\left(\dot{r}\ddot{r} + \omega\dot{\omega}r^2 + \omega^2 r\dot{r}\right)}{c^2\left(1 - \frac{\dot{r}^2}{c^2} - \frac{\omega^2 r^2}{c^2}\right)^{3/2}} \tag{5.4}$$

Substitution of (5.3) and (5.4) into (5.2) then yields after reduction, ($\mathbf{F}$ is now purely spatial),

$$\boldsymbol{F} = \frac{m_o}{\left(1-\frac{\dot{r}^2}{c^2}-\frac{\omega^2 r^2}{c^2}\right)^{3/2}}\left[\begin{array}{l}\left\{\left(\ddot{r}-\omega^2 r\right)\left(1-\frac{\omega^2 r^2}{c^2}\right)+\left(2\omega\dot{r}+\dot{\omega}r\right)\frac{\omega r\dot{r}}{c^2}\right\}\boldsymbol{n}\\ +\left\{\left(2\omega\dot{r}+\dot{\omega}r\right)\left(1-\frac{\dot{r}^2}{c^2}\right)+\left(\ddot{r}-\omega^2 r\right)\frac{\omega r\dot{r}}{c^2}\right\}\boldsymbol{t}\end{array}\right] \tag{5.5}$$

This is the most general form of the force equation for spatially accelerated curvi - linear motion in a plane in $\mathbf{D_0}$ and which clearly possesses a distinct symmetry.

<u>The Case of a Purely Radial Force</u>

If $\boldsymbol{F}$ is purely radial (constant angular momentum), then in (5.2), in addition to the temporal component, the radial normal component will also be zero. Thus

$$m\left(2\omega\dot{r} + \dot{\omega}r\right) = -\dot{m}\omega r \tag{5.6}$$

which from (5.3) and (5.4) becomes

$$2\omega\dot{r} + \dot{\omega}r = -\frac{\omega r\,\dot{r}\left(\ddot{r} - \omega^2 r\right)}{c^2\left(1 - \frac{\dot{r}^2}{c^2}\right)} \tag{5.7}$$

which can also be obtained from the radial normal component of (5.5). Substitution of (5.7) into (5.5) gives

$$\boldsymbol{F} = \frac{m_o\left(\ddot{r} - \omega^2 r\right)\boldsymbol{n}}{\left(1 - \frac{\dot{r}^2}{c^2} - \frac{\omega^2 r^2}{c^2}\right)^{1/2}\left(1 - \frac{\dot{r}^2}{c^2}\right)} \tag{5.8}$$

This is the equation of motion of a point mass in a plane in $\mathbf{D_0}$ subjected to an arbitrary spatial radial force. Note that substitution of (5.7) into (5.4) yields after reduction

$$\dot{m} = \frac{m_o\dot{r}\left(\ddot{r} - \omega^2 r\right)}{c^2\left(1 - \frac{\dot{r}^2}{c^2} - \frac{\omega^2 r^2}{c^2}\right)^{1/2}\left(1 - \frac{\dot{r}^2}{c^2}\right)} \tag{5.9}$$

which when substituted into (5.8) gives

$$\boldsymbol{F} = \frac{\dot{m}c^2}{\dot{r}}\boldsymbol{n} \tag{5.10}$$

This is clearly seen to be identical to the rectilinear case and provides further confirmation that the mass rate effect only exists along coincident elements of the force and velocity vectors.

Conversion of the Equation of Motion to Proper Time

To determine the equation of the orbit it is first necessary to convert the equation of motion to the proper time of the point mass.

Conversion of (5.8) to the proper time of the point mass, is achieved as follows. With

$$\frac{dt_p}{dt} = \left(1 - \frac{\dot{r}^2}{c^2} - \frac{\omega^2 r^2}{c^2}\right)^{1/2} \tag{5.11}$$

then

$$\frac{dr}{dt_p} = \dot{r}\frac{dt}{dt_p} = \frac{\dot{r}}{\left(1 - \frac{\dot{r}^2}{c^2} - \frac{\omega^2 r^2}{c^2}\right)^{1/2}} \tag{5.12}$$

consequently with

$$\frac{d^2r}{dt_p^2} = d\left(\frac{dr}{dt_p}\right)/dt_p = \left[d\left(\frac{dr}{dt_p}\right)/dt\right]\frac{dt}{dt_p} \tag{5.13}$$

Substitution from (5.11) and (5.12) gives

$$\frac{d^2r}{dt_p^2} = \frac{\ddot{r}\left(1 - \frac{\omega^2 r^2}{c^2}\right) + \frac{\omega\dot{\omega}r^2\dot{r}}{c^2} + \frac{\omega^2 r\,\dot{r}^2}{c^2}}{\left(1 - \frac{\dot{r}^2}{c^2} - \frac{\omega^2 r^2}{c^2}\right)} \tag{5.14}$$

But from (5.7)

$$\frac{\omega\dot{\omega}r^2\dot{r}}{c^2} = -\frac{2\omega^2 r\,\dot{r}^2}{c^2} - \frac{\omega^2 r^2\dot{r}^2\left(\ddot{r} - \omega^2 r\right)}{c^4\left(1 - \frac{\dot{r}^2}{c^2}\right)} \tag{5.15}$$

Which, when substituted into (5.14) gives after reduction

$$\frac{d^2r}{dt_p^2}=\frac{\ddot{r}-\dfrac{\omega^2 r\dot{r}^2}{c^2}}{\left(1-\dfrac{\dot{r}^2}{c^2}\right)\left(1-\dfrac{\dot{r}^2}{c^2}-\dfrac{\omega^2 r^2}{c^2}\right)} \tag{5.16}$$

Also from (5.8) after taking the magnitude

$$\ddot{r}=\omega^2 r+\left(1-\frac{\dot{r}^2}{c^2}-\frac{\omega^2 r^2}{c^2}\right)^{1/2}\left(1-\frac{\dot{r}^2}{c^2}\right)\frac{F}{m_0} \tag{5.17}$$

and substitution of this into (5.16) then yields

$$\frac{d^2r}{dt_p^2}=\frac{F}{m_0\left(1-\dfrac{\dot{r}^2}{c^2}-\dfrac{\omega^2 r^2}{c^2}\right)^{1/2}}+\frac{\omega^2 r}{\left(1-\dfrac{\dot{r}^2}{c^2}-\dfrac{\omega^2 r^2}{c^2}\right)} \tag{5.18}$$

but from (5.11)

$$\omega=\frac{d\varphi}{dt}=\left(\frac{d\varphi}{dt_p}\right)\left(\frac{dt_p}{dt}\right)=\omega_p\left(1-\frac{\dot{r}^2}{c^2}-\frac{\omega^2 r^2}{c^2}\right)^{1/2} \tag{5.19}$$

so that this gives in (5.18)

$$\frac{d^2r}{dt_p^2}=\frac{F}{m_0\left(1-\dfrac{\dot{r}^2}{c^2}-\dfrac{\omega^2 r^2}{c^2}\right)^{1/2}}+\omega_p^2 r \tag{5.20}$$

It now only remains to convert the term $\left(1-\dot{r}^2/c^2-\omega^2 r^2/c^2\right)^{1/2}$ to proper time as follows; rewriting (5.12) as

$$\dot{r}_p=\frac{\dot{r}}{\left(1-\dfrac{\dot{r}^2}{c^2}-\dfrac{\omega^2 r^2}{c^2}\right)^{1/2}} \tag{5.21}$$

and, from (5.19), with

$$\omega_p r=\frac{\omega r}{\left(1-\dfrac{\dot{r}^2}{c^2}-\dfrac{\omega^2 r^2}{c^2}\right)^{1/2}} \tag{5.22}$$

then

$$\dot{r}_p^2 + \omega_p^2 r^2 = \frac{\dot{r}^2 + \omega^2 r^2}{\left(1 - \frac{\dot{r}^2}{c^2} - \frac{\omega^2 r^2}{c^2}\right)} \tag{5.23}$$

and rearrangement of this then gives

$$\left(1 - \frac{\dot{r}^2}{c^2} - \frac{\omega^2 r^2}{c^2}\right)^{1/2} = \left(1 + \frac{\dot{r}_p^2}{c^2} + \frac{\omega_p^2 r^2}{c^2}\right)^{-1/2} \tag{5.24}$$

so that (5.20) finally becomes

$$\frac{d^2 r}{dt_p^2} = \omega_p^2 r + \left(1 + \frac{\dot{r}_p^2}{c^2} + \frac{\omega_p^2 r^2}{c^2}\right)^{1/2} \frac{F}{m_0} \tag{5.25}$$

and for a purely spatial radial force, is the equation of planar motion in $\mathbf{D_0}$ expressed as a function of the proper time.

Derivation of the Equation of the Orbit

To obtain the equation of the orbit from (5.25), it is initially necessary to evaluate the first integral of (5.6). Rearrangement of that equation yields

$$-\frac{\dot{m}}{m} - 2\frac{\dot{r}}{r} - \frac{\dot{\omega}}{\omega} = 0 \tag{5.26}$$

Integrating (5.26) gives

$$\frac{m_0 \omega r^2}{\left(1 - \frac{\dot{r}^2}{c^2} - \frac{\omega^2 r^2}{c^2}\right)^{1/2}} = k \tag{5.27}$$

which then, after the determination of the constant of integration, k from initial conditions, $r = r_0$ and $\omega = \omega_0$, when $\dot{r} = 0$ gives, together with (5.22)

$$\omega_p r^2 = \frac{\omega_0 r_0^2}{\left(1 - \frac{\omega_0^2 r_0^2}{c^2}\right)^{1/2}} \tag{5.28}$$

and in line with convention this constant is designated, h. The equation of the orbit may now be obtained in the usual way thus.

Putting

$$r = \frac{1}{\mu} \tag{5.29}$$

then

$$\frac{dr}{dt_p} = -\frac{\omega_p}{\mu^2}\frac{d\mu}{d\varphi} = -h\frac{d\mu}{d\varphi} \tag{5.30}$$

and

$$\frac{d^2r}{dt_p^2} = -\left(h\omega_p\right)\left(\frac{d^2\mu}{d\varphi^2}\right) = -\left(h^2\mu^2\right)\left(\frac{d^2\mu}{d\varphi^2}\right) \tag{5.31}$$

Insertion of (5.28), (5.29) and (5.30) and (5.31) into (5.25) then gives the desired result for the equation of the orbit.

$$\frac{d^2\mu}{d\varphi^2} + \mu = -\frac{F}{m_0 h^2\mu^2}\left[1 - \left\{\mu^2 + \left(\frac{d\mu}{d\varphi}\right)^2\right\}\frac{h^2}{c^2}\right]^{1/2} \tag{5.32}$$

Solution of the Equation of the Orbit for Two Oppositely Charged Particles In a Vacuum.

Assuming conditions are such that the only effect between the two particles is an electrostatic one and that their relative size is such that the smaller has negligible effect upon the larger, and ignoring any spin effects, then the force of attraction between them may be expressed as

$$F = F_0\mu^2 \tag{5.33}$$

The equation of the orbit of the smaller particle, from (5.32) and (5.33) then becomes

$$\frac{d^2\mu}{d\varphi^2} + \mu = \frac{F_0}{m_0 h^2}\left[1 - \left\{\mu^2 + \left(\frac{d\mu}{d\varphi}\right)^2\right\}\frac{h^2}{c^2}\right]^{1/2} \tag{5.34}$$

to solve this equation put

$$\mu^2 + \left(\frac{d\mu}{d\varphi}\right)^2 = \Theta^2 \tag{5.35}$$

this being the inverse of the perpendicular distance from a focal point of the orbit to a tangent at any point on the spatial trajectory. Differentiating (5.35) with respect to φ gives

$$\frac{d^2\mu}{d\varphi^2} + \mu = \Theta\frac{d\Theta}{d\mu} \tag{5.36}$$

Substitution of (5.35) and (5.36) into (5.34) then yields

$$\Theta\frac{d\Theta}{d\mu} = \left(1 - \frac{h^2\Theta^2}{c^2}\right)^{1/2}\frac{F}{m_0 h^2} \tag{5.37}$$

This equation can now be solved using standard methods to yield

$$(\mu-\mu_0)\frac{F_0}{m_0c^2}=-\left[1-\left\{\mu^2+\left(\frac{d\mu}{d\varphi}\right)^2\right\}\frac{h^2}{c^2}\right]^{1/2}+\left(1-\frac{h^2\mu_0^2}{c^2}\right)^{1/2} \tag{5.38}$$

where the constant of integration has, together with (5.35), been inserted. Rearranging (5.38) for $d\mu/d\varphi$ gives

$$\left(\frac{d\mu}{d\varphi}\right)^2=-\left(1+\frac{F_0^2}{m_0^2c^2h^2}\right)\mu^2 -2\left(\frac{F_0}{m_0h^2}\right)\left\{\left(1-\frac{h^2\mu_0^2}{c^2}\right)^{1/2}+\frac{F_0\mu_0}{m_0c^2}\right\}\mu +\mu_0^2\left(1-\frac{F_0^2}{m_0^2c^2h^2}\right)-2\left(\frac{F_0{}^2\mu_0}{m_0h^2}\right)\left(1-\frac{h^2\mu_0^2}{c^2}\right)^{1/2} \tag{5.39}$$

This equation is also a standard type that can be solved using conventional methods to yield, after some reduction

$$\mu=\left[\frac{\left(\frac{F_0}{m_0h^2}\right)\left\{\left(1-\frac{h^2\mu_0^2}{c^2}\right)^{1/2}+\frac{F_0\mu_0}{m_0c^2}\right\}}{\left(1+\frac{F_0^2}{m_0^2c^2h^2}\right)}\right]\left[1+\frac{\left\{\frac{m_0h^2\mu_0}{F_0}-\left(1-\frac{h^2\mu_0^2}{c^2}\right)^{1/2}\right\}\boldsymbol{\cos\Phi}}{\left\{\frac{F_0\mu_0}{m_0c^2}+\left(1-\frac{h^2\mu_0^2}{c^2}\right)^{1/2}\right\}}\right] \tag{5.40}$$

where

$$\Phi=\frac{\varphi}{\left(1+\frac{F_0^2}{m_0^2c^2h^2}\right)^{1/2}} \tag{5.41}$$

and where initial conditions have been chosen such that the constant of integration is zero. Equation (5.40) describes the spatial trajectory of the smaller particle about the larger and clearly, as in the literature [5], is seen to be a rotating conic section. From the second part of (5.40), this rotation is seen to be a function of the finite Spatial Terminal Velocity, c, within $\mathbf{D_0}$, and also, that the precession angle is a constant retrograde one being, unlike the gravitational case, independent of the term μ.

6. Concluding Remarks

The concept of Existence Velocity within the Relativistic Domain $\mathbf{D_0}$, both as have been defined in this paper, has, using methods identical to those of classical mechanics, enabled a simplified mathematical formulation of the kinematics and kinetics of the Special Theory of Relativity, for one particular case, rectilinear motion. In doing so, it has also provided a clearer insight into the relativistic nature of an applied spatial force, and the associated kinetic energy that it produces in a spatially accelerated point mass. In Sections 4 and 5 this simplified method has been used to examine two further relativistic kinematic situations, linear planar, and central orbital motion, as demonstrations of its application.
It must be noted that these concepts are only valid within the domain $\mathbf{D_0}$ i.e. Pseudo-Euclidean Space - Time, (see Appendix A). For other relativistic domains, such as one containing gravitation, it is necessary to change the characteristics of the Domain accordingly. This will be the subject of the next paper where a new theroem for gravitation will be presented using a suitably modified Relativistic Domain. This new theroem will differ from that of the General Theory of Relativity in that although the modified Domain will differ from Pseudo-Euclidean Space-Time, it will still be one exhibiting a linear co-ordinate system.

One final note. In the Special Theory, the limiting spatial velocity is generally accepted to be the velocity of electromagnetic radiation in a vacuum, i.e. light. In this paper the limiting velocity has been termed the "Spatial Terminal Velocity" and no reference has been made to the velocity of light. This approach has been adopted because it has not been conclusively proven that the limiting spatial velocity in the Special Theory is indeed the velocity of light. It may well be that the true limiting velocity is slightly different from this. This is believed possible because from the Special Theory it is known that matter possessing mass and thus energy, cannot be accelerated to the velocity of light within a finite time. Yet it is also known that electromagnetic radiation possesses energy and must therefore also possess mass, however small. Thus it is considered probable that the velocity of light may well be slightly lower than the true limiting spatial velocity in the Special Theory and therefore also in $\mathbf{D_0}$.

APPENDIX A

Equivalence of the Domain $\mathbf{D_0}$ with Pseudo - Euclidean Space - Time

To fully reconcile the application of the concepts presented in this paper with the Special Theory of Relativity, it is necessary to demonstrate that the Domain $\mathbf{D_0}$ is equivalent to the space - time in which that theory applies, Pseudo - Euclidean Space - Time. It is therefore necessary to show that $\mathbf{D_0}$ possesses the following additional characteristics to those already defined.

(i) The Temporal Co-ordinate $\mathbf{X}_o$ is related to the time t in $\mathbf{D_0}$ by the expression, (after Minkowski),

$$\mathbf{X}_0 = \boldsymbol{j}ct \tag{A.1}$$

(ii) The magnitude of the maximum theoretically attainable spatial velocity in $\mathbf{D_0}$ is equal to the Spatial Terminal Velocity, c.

(iii) When the spatial velocity of a moving point in $\mathbf{D_0}$ is rectilinear and constant, measurements of time and distance related to axes associated with it, transform from those of $\mathbf{D_0}$ according to the Lorentz Transformations of the Special Theory.

The Temporal Co-Ordinate $\mathbf{X_0}$

In (2.7), if $\boldsymbol{v}$ is zero, i.e. a point is spatially at rest in $\mathbf{D_0}$, its Existence Velocity is reduced to:

$$\mathbf{V} = \boldsymbol{j}c \tag{A.2}$$

which upon integration with respect to t gives:

$$\mathbf{S} = \boldsymbol{r}_0 + \boldsymbol{j}ct \tag{A.3}$$

where $\boldsymbol{r}_o$, the constant of integration, is the constant spatial position in $\mathbf{D_0}$ from some stationary reference. In this case the trajectory of motion is clearly, from (A.2), along the temporal axis $\mathbf{X_0}$ so that, in (A.3), the relationship of (A.1) is implicit.

The Maximum Theoretically Attainable Spatial Velocity in $\mathbf{D_0}$.

Inserting (2.8) and (2.9) into (2.7) gives for Existence Velocity in spatial - temporal polar form

$$\mathbf{V} = \boldsymbol{s}cSin\theta + \boldsymbol{j}cCos\theta \tag{A.4}$$

where $\boldsymbol{s}$ is a unit vector in the direction of $\boldsymbol{v}$. Clearly the maximum theoretically attainable spatial velocity occurs when $\theta = \pi/2$, the Existence Velocity becoming simply:

$$\mathbf{V} = \boldsymbol{s}c \tag{A.5}$$

At any other value of θ, the spatial component of $\mathbf{V}$ must be less than c.

Transformation of The Axes

This is the more complex of the three criteria to prove. It is accomplished by derivation from first principles, of the relationship between the spatial and temporal axes of a point in $\mathbf{D_0}$ moving with constant velocity, and those of $\mathbf{D_0}$ itself. The temporal axis relationship is derived in the form of a time parameter in order to fully demonstrate agreement with the Lorentz Transformations. During this process a relationship for the proper time of the point is also derived.

Let B be a point in $\mathbf{D_0}$ moving with a constant spatial rectilinear velocity $\boldsymbol{v}$. Let $\mathbf{R'}$ be a "space like" co-ordinate associated with B, let Q be any fixed point on $\mathbf{R'}$, and let t'_q be the time along a "temporal" axis $\mathbf{X'_q}$, associated with $\mathbf{R'}$ at the location of Q. Finally, let initial conditions be such that at some instant in $\mathbf{D_0}$ designated $t = 0$, the position of B in $\mathbf{D_0}$ is defined as a reference point, (Fig. A1 may be usefully referred to in following this derivation).

Consider first the spatial axis $\mathbf{R'}$. At time t the positions of B and Q in $\mathbf{D_0}$ will be given by :

$$\mathrm{S}_b = \boldsymbol{r_b} + \boldsymbol{j}\,ct_b \tag{A.6}$$

$$\mathrm{S}_q = \boldsymbol{r_q} + \boldsymbol{j}\,ct_q \tag{A.7}$$

From (A.6) and (A.7) the position of Q on $\mathbf{R'}$ can be expressed in spatial - temporal vector form as:

$$\mathbf{r}'_q = \mathbf{S_q} - \mathbf{S_b} = \boldsymbol{r}_q - \boldsymbol{r}_b + \boldsymbol{j}\,c\left(t_q - t_b\right) \tag{A.8}$$

Differentiating (A.8) with respect to t

$$\frac{d\mathbf{r}'_\mathbf{q}}{dt} = \frac{d\,\boldsymbol{r_q}}{dt} - \frac{d\,\boldsymbol{r}_b}{dt} + \boldsymbol{j}\left(\frac{dt_q}{dt} - \frac{dt_b}{dt}\right) \tag{A.9}$$

But with

$$\frac{d\,\boldsymbol{r_b}}{dt} = \boldsymbol{v} \tag{A.10}$$

and

$$\frac{dt_b}{dt} = \left(1 - \frac{v^2}{c^2}\right)^{1/2} \tag{A.11}$$

In (A.9) this gives

$$\frac{d\mathbf{r}'_q}{dt} = \frac{d\,\boldsymbol{r_q}}{dt} - \boldsymbol{v} + \boldsymbol{j}\left(\frac{dt_q}{dt} - \left(1 - \frac{v^2}{c^2}\right)^{1/2}\right) \tag{A.12}$$

and taking the magnitude of (A.12)

$$\frac{dr'_q}{dt} = \left[\left(\frac{dr_q}{dt} - v\right)^2 + c^2\left(\frac{dt_q}{dt} - \left(1 - \frac{v^2}{c^2}\right)^{1/2}\right)^2\right]^{1/2} \tag{A.13}$$

but for Q to exist in $\mathbf{D_0}$

$$\left(\frac{dr_q}{dt}\right)^2+c^2\left(\frac{dt_q}{dt}\right)^2=c^2 \tag{A.14}$$

so that in (A.13)

$$\frac{dr_q'}{dt}=\left[2c^2-2v\frac{dr_q}{dt}-2c^2\left(1-\frac{v^2}{c^2}\right)^{1/2}\left(\frac{dt_q}{dt}\right)\right]^{1/2} \tag{A.15}$$

but r'_q is constant, therefore dr'_q/dt must be zero. This gives in (A.15)

$$\frac{dt_q}{dt}=\frac{1-\left(\frac{v}{c^2}\right)\frac{dr_q}{dt}}{\left(1-\frac{v^2}{c^2}\right)^{1/2}} \tag{A.16}$$

As v is constant (A.16) can be integrated immediately to give:

$$t_q=\frac{t-r_q\frac{v}{c^2}}{\left(1-\frac{v^2}{c^2}\right)^{1/2}}+k \tag{A.17}$$

Substitution of (A.17) into (A.8) then yields:

$$\mathbf{r}_q'=\boldsymbol{r}_q-\boldsymbol{v}t-\boldsymbol{j}c\left[\frac{v(r_q-vt)}{c^2\left(1-\frac{v^2}{c^2}\right)^{1/2}}-k\right] \tag{A.18}$$

Now as Q is any point on the $\mathbf{R}'$ axis, it can be co-incident with B to give $\boldsymbol{r}_q'=0$. In this case (A.18) would reduce to:

$$\boldsymbol{r_q}=\boldsymbol{v}t \quad and \quad k=0 \tag{A.19}$$

Therefore k must be zero for all $\mathbf{R}'$ Consequently (A.17) and (A.18) respectively reduce to:

$$t_q=\frac{t-r_q\frac{v}{c^2}}{\left(1-\frac{v^2}{c^2}\right)^{1/2}} \tag{A.20}$$

and

$$\mathbf{r'_q} = \boldsymbol{r}_q - \boldsymbol{v}\,t - \boldsymbol{j}\left(\frac{v(r_q - vt)}{c\left(1-\frac{v^2}{c^2}\right)^{1/2}}\right) \qquad \text{(A.21)}$$

The magnitude of (A.21) yielding:

$$r'_q = \frac{r_q - vt}{\left(1-\frac{v^2}{c^2}\right)^{1/2}} \qquad \text{(A.22)}$$

As Q is any point on $\mathbf{R}'$, then (A.21) and (A.22) represent the relationship between the spatial axis associated with B and that of $\mathbf{D_0}$ and (A.20) represents a measure of the proper time of Q in $\mathbf{D_0}$. The parameters $\mathbf{r'_q}$ and $\boldsymbol{r_q}$ may therefore, in (A.21) and (A.22), be replaced with the axis designators $\mathbf{R}'$, and $\mathbf{R}$ respectively, and the lengths r'_q. and r_q therefore represent the relationship between their scales. Subsequent reference to (A.21) shows that $\mathbf{R}'$ possesses both spatial and temporal components and therefore a precise orientation in $\mathbf{D_o}$. Substitution of (A.22), (2.8) and (2.9) into (A.21) gives:

$$\mathbf{R'} = \mathrm{R}'(\boldsymbol{s}\,\boldsymbol{cos}\,\theta - \boldsymbol{j}\,\boldsymbol{sin}\,\theta) \qquad \text{(A.23)}$$

Where $\boldsymbol{s}$ is a unit vector in the direction of $\boldsymbol{v}$. This shows by comparison with (A.4) that $\mathbf{R}'$ is orthogonal to the Existence Velocity Vector and, therefore, the spatial - temporal trajectory of B in $\mathbf{D_0}$. Also, from (A.22) it is clear that units of length along $\mathbf{R'}$ (i.e. with t constant) are greater than units of length along $\mathbf{R}$, and that the increase is a direct result of the orientation of $\mathbf{R}'$ relative to $\mathbf{R}$ in $\mathbf{D_0}$.

Now consider the temporal co-ordinate associated with $\mathbf{R}'$ at the location of Q. Firstly it is noted that since Q is fixed in relation to B, its only motion in axes associated with B is a temporal one. Therefore the primed temporal axis along which Q is in motion, $\mathbf{X'_q}$, must lie along its trajectory in $\mathbf{D_0}$ which must be parallel to that of B. As a consequence, this axis must be orthogonal to $\mathbf{R}'$. Now (A.20), as stated above, is a measure of the proper time of Q in $\mathbf{D_0}$ and by virtue of (A.7) is therefore directly proportional to its position on $\mathbf{X}_0$ from its initial position at $t = o$. In like manner however, the proper time of Q on its primed temporal axis is directly proportional to its position on that axis from its initial position. Note however that, as Q possesses only temporal motion in the primed axes, its proper time along $\mathbf{X'_q}$ will be identical to that of $\mathbf{X'_q}$ itself. Therefore, to derive the relationship between time on the two temporal axes for any constant value of r_q, a one-to-one correspondence between incremental distances on them can be established as follows. If $d\mathbf{X_0}$ is an incremental distance along the temporal axis of $\mathbf{D_0}$, and $d\mathbf{X'_q}$ an incremental distance along the temporal axis associated with the point Q on $\mathbf{R}'$ such that they are temporily coincident in $\mathbf{D_0}$, then due to their relative orientation, they conform to the following expression:

$$d\mathbf{X}'_{\mathbf{q}} = \frac{d\mathbf{X}_{\mathbf{0}}}{\left(1-\frac{v^2}{c^2}\right)^{1/2}} \qquad \text{(A.24)}$$

As t'_q is the time along $\mathbf{X'}_q$, and since the primed temporal velocity of all points on $\mathbf{R'}$ is c, i.e. equal to $|\mathbf{V}|$, then by (A.2), (A.24) may be rewritten thus:

$$dt'_q = \frac{dt}{\left(1-\frac{v^2}{c^2}\right)^{1/2}} \qquad \text{(A.25)}$$

Integrating (A.25)

$$t'_q = \frac{t}{\left(1-\frac{v^2}{c^2}\right)^{1/2}} + k \qquad \text{(A.26)}$$

The constant k is the initial condition that ensures temporal coincidence of the two incrementals within $\mathbf{D_0}$, and k must therefore be such that when t'_q is zero, the proper time of Q in $\mathbf{D_0}$ is also zero. Thus, from (A.20) when t_q is zero, t is given by

$$t = r_q \frac{v}{c^2} \qquad \text{(A.27)}$$

which gives in (A.26) when t'_q is zero

$$k = \frac{-r_q \frac{v}{c^2}}{\left(1-\frac{v^2}{c^2}\right)^{1/2}} \qquad \text{(A.28)}$$

so that finally in (A.26)

$$t'_q = \frac{t - r_q \frac{v}{c^2}}{\left(1-\frac{v^2}{c^2}\right)^{1/2}} \qquad \text{(A.29)}$$

This being the relationship between time on the $\mathbf{X'_q}$ and $\mathbf{X}_o$ axes.

Note that (A 25) shows that the units of time along $\mathbf{X'}_q$ are greater than those along $\mathbf{X}_o$ by the same factor and, for the same reason that the units of length along the spatial axes differ. Clearly all such points on $\mathbf{R'}$, including B, must have associated with them a time, along a unique temporal axis, of the form of (A.29) in which the spatial term differs appropriately. The locus of the reference zero on these axes lies along the spatial axis of $\mathbf{D_0}$ at $t = 0$. This together with the expansion of the units of time along these axes ensures the simultaneity of existence of each point on $\mathbf{R'}$ in both frames of reference. It is also noted that the mathematical relationship between t'_q and t is the same as that between t_q and t. They cannot be equated however because of the difference in the magnitude of the units.

The above relationships, specifically associated with the point Q, can be diagrammatically represented as in Fig. (A1) below. The spatial motion of the point B, has, from the above argument, resulted in axes associated with it being expanded and rotated in the direction of motion in $\mathbf{D_0}$ through the same spatial-temporal angle θ as the Existence Velocity vector of B. This concurs with statements in the literature [3] that in Minkowski's `World' the Lorentz Transformations "correspond to a rotation of the co-ordinate system".

Clearly (A.22) and (A.29) are identical to the Lorentz Transformations of the Special Theory and together with the previous results of this Appendix, demonstrates the equivalence of the Domain $\mathbf{D_0}$ with Pseudo - Euclidean Space - Time. The application of the concept of Existence Velocity within the latter is therefore a valid one.

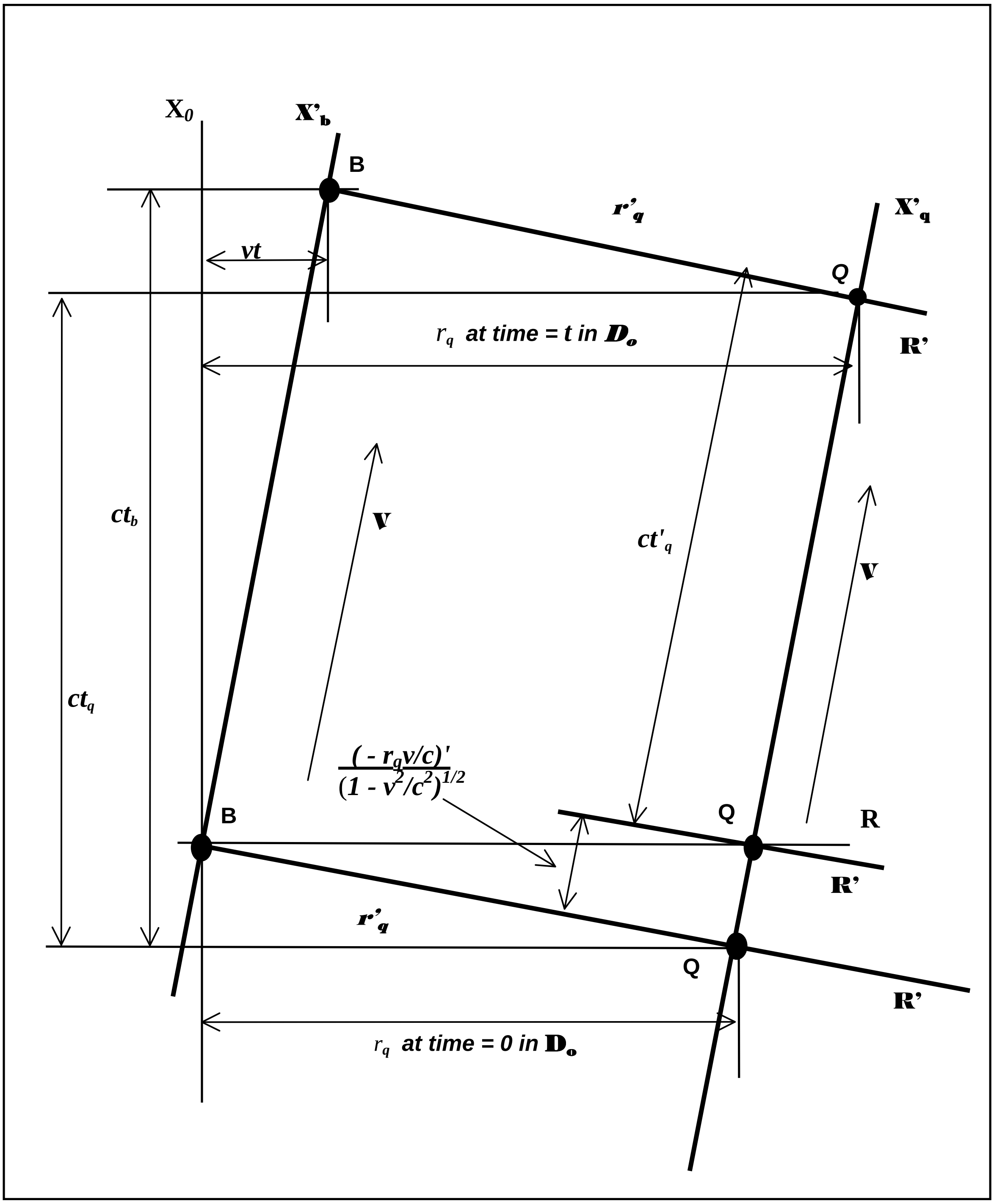

FIG. A.1: DIAGRAMMATIC REPRESENTATION OF THE RELATIONSHIP BETWEEN THE REFERENCE AXES OF $\mathbf{D_o}$ AND THOSE ASSOCIATED WITH B AT THE POINT Q

APPENDIX B

Reduction of Selected Relativistic Equations to their Classical Equivalents

In all cases this is effected by allowing the constant velocity parameter c to become infinite. Only the main equations for which a classical equivalent exists is so treated. Trivial examples will be ignored unless a special condition is implied.

Section 2

(i) Eq. (2.6), the temporal rate. When $c \to \infty$

$$\frac{dt_p}{dt} = 1 \tag{B.1}$$

Hence in classical theory the proper time of a moving body is identical to the time in $\mathbf{D_0}$, Pseudo-Euclidean Space-Time.

(ii) Eq. (2.7), Existence Velocity. When $c \to \infty$

$$\mathrm{V} = \boldsymbol{v} + \boldsymbol{j}\infty \tag{B.2}$$

Hence temporal velocity in classical studies is "infinite". Existence velocity does not exist in classical mechanics, and temporal velocity in such studies, is therefore a meaningless concept because it implies that time passes infinitely quickly. Where a concept does not exist in classical mechanics etc, relativistic reduction generally results in an infinite or zero value.

Section 3.

(iii) Eqs. (3.6) and (3.9), Rest, Energy and Inertial Mass. When $c \to \infty$

$$m_0 = m = m_a \tag{B.3}$$

Thus rest, energy and inertial mass are identical in classical mechanics. Hence any reference to inertial mass in such studies is meaningless. Consequently, as is evident from Eq (3.7), when $c \to \infty$

$$\frac{dm}{dt} = 0 \tag{B.4}$$

(iv) Eq (3.19), Kinetic Energy. To determine the classical expression for kinetic energy directly from (3.19) would be incorrect because (3.19) is a relationship in matter energy, a concept that does not exist in classical mechanics. The correct procedure is first to insert (3.6) into (3.19) and expand the result binomially to yield

$$E_k = \frac{m_0 v^2}{2} + \frac{3m_0 v^4}{8c^2} + \frac{15m_0 v^6}{48c^4} + \ldots \tag{B.5}$$

from which, when $c \to \infty$

$$E_k = \frac{m_0 v^2}{2} \tag{B.6}$$

the classical result.

Section 4.

(v) Eq. (4.19) and (4.20), the acceleration vectors along the co-ordinant axes, when $c \to \infty$

$$\frac{dv_x}{dt} = \frac{F\, \boldsymbol{cos}\,\xi}{m_0} \tag{B.7}$$

$$\frac{dv_y}{dt} = \frac{F\, \boldsymbol{sin}\,\xi}{m_0} \tag{B.8}$$

and clearly therefore the force and acceleration vectors are coincident.

(vi) Eq. (4.21) and (4.22), the mass on each co-ordinant axis, when $c \to \infty$

$$m_{ax} = m_{ay} = m_0 \tag{B.9}$$

(vii) Eq. (4.24), Angular relationship between the acceleration vector and the $\mathbf{X}$ axis, when $c \to \infty$

$$\boldsymbol{tan}\,\Psi = \boldsymbol{tan}\,\eta \tag{B.10}$$

which confirms the result at (B.7) and (B.8).

(viii) Eqs. (4.25) and (4.28), accelerations along and normal to the velocity vector, when $c \to \infty$.

$$\frac{dv}{dt} = \frac{F}{m_0}\boldsymbol{cos}(\xi - \eta) \tag{B.11}$$

and

$$\frac{dv_n}{dt} = \frac{F}{m_0}\boldsymbol{sin}(\xi - \eta) \tag{B.12}$$

so that from (B.7), (B.8), (B.11) and (B12)

$$\left[\left(\frac{dv_x}{dt}\right)^2 + \left(\frac{dv_y}{dt}\right)^2\right]^{1/2} = \left[\left(\frac{dv}{dt}\right)^2 + \left(\frac{dv_n}{dt}\right)^2\right]^{1/2} = \frac{F}{m_0} \tag{B.13}$$

which now also shows that the force, acceleration and velocity vectors are co-incident.

Section 5.

(ix) Eq. (5.5), general curvi-linear equation of motion, when $c \to \infty$

$$F = m_0\left[\left(\ddot{r} - \omega^2 r\right)\boldsymbol{n} + \left(2\omega\dot{r} + \dot{\omega}r\right)\boldsymbol{t}\right] \tag{B.14}$$

the classical equation in mechanics.

(x) Eq. (5.25), equation of planar motion in the proper time, when $c \to \infty$

$$\frac{d^2r}{dt^2} = \frac{F}{m_0} + \omega^2 r \tag{B.15}$$

the classical equation in mechanics.

(xi) Eq. (5.32), the equation of a central orbit, when $c \to \infty$

$$\frac{d^2\mu}{d\varphi^2} + \mu = \frac{F_0}{m_0 h^2 \mu^2} \tag{B.16}$$

where

$$h = \omega_0 r_0^2 \tag{B.17}$$

the classical equations in mechanics.

(xii) Eq. (5.40), the equation of a central orbit trajectory, when $c \to \infty$

$$\mu = \frac{F_0}{m_0 h^2}\left[1 + \left(\frac{m_0 h^2 \mu_0}{F_0} - 1\right)\boldsymbol{cos}\,\varphi\right] \tag{B.18}$$

which is clearly the equation of a simple conic section. The semi-latus rectum and eccentricity are

$$L = \frac{m_0 h^2}{F_0} \quad \text{and} \quad e = \frac{m_0 h^2 \mu_0}{F_0} - 1 \tag{B.19}$$

These are again the classical results.

A Belated Acknowlegement.
Since the first publication of this paper, it has been learned that in his excellent book, "Relativity Visualised", Lewis Carrol Epstein, used a concept very similar to that in this paper to explain the impossibility of travelling at the speed of light.

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