THE RIEMANN HYPOTHESIS

-

A RESOLUTION.

Peter G.Bass.

M9 Ver 1.0.0
www.relativitydomains.com
January 2013

ABSTRACT.

This paper provides a simple resolution to the Riemann Hypothesis concerning the roots of the Riemann Zeta Function.

CONTENTS.

1.0 Introduction.

The Riemann Hypothesis has remained an intractable but fascinating problem since its proposal by Bernhard Riemann in 1859. It has been studied extensively since then, using procedures derived from the manipulation of certain infinite series, but without a final result being achieved. It is believed that this is due to the fact that, such manipulation is based upon a false premise concerning how infinite series can be mathematically treated. Consequently, the approach taken here does not use these methods, but instead employs a very simple elementary development, using nothing more than the relationship between Riemann's Zeta Function, and the circular Sine and Cosine Functions. This leads very quickly to a solution showing that the Riemann Hypothesis is false. This is the subject of the first part of this paper.

Because this result is in total contradiction to current belief, and the classical results determined from the methods briefly mentioned above, the second part of this paper reviews those methods for their veracity, and shows how the false premise referred to above has distorted the results achieved.

Note that this paper will employ the word "infinity" and the symbols "+ ∞" and "- ∞" but that such use does not imply they are numbers, but instead, means "to increase without limit".

2.0 The Riemann Hypothesis - A Resolution.

The Riemann Hypothesis as effectively stated by Bernhard Riemann is, "The zeros of the Zeta Function all have real part 1/2."

Express the Zeta Function as follows

$$\varsigma(s) = 1 + \sum_{N=2} \frac{1}{N^s} \tag{2.1}$$

Let $s = a + jb$ where a and b are real but not zero, then

$$\varsigma(s) = 1 + \sum_{N=2} \frac{1}{N^{(a+jb)}} \tag{2.2}$$

Separating the real and imaginary components

$$\varsigma(s) = 1 + \sum_{N=2} \frac{1}{N^a N^{jb}} \tag{2.3}$$

Moving the imaginary component to the numerator

$$\varsigma(s) = 1 + \sum_{N=2} \frac{N^{-jb}}{N^a} \tag{2.4}$$

Now consider the numerator

$$f(N) = N^{-jb} \tag{2.5}$$

Taking Naperian logarithms

$$Ln[f(N)] = -jbLn(N) \tag{2.6}$$

so that

$$f(N) = e^{Ln[f(N)]} = e^{-jbLn(N)} \tag{2.7}$$

which can be expressed as

$$f(N) = \boldsymbol{cos}[-bLn(N)] \; + j\,\boldsymbol{sin}[-bLn(N)] \tag{2.8}$$

Substituting this back into (2.4), adjusting signs appropriately gives

$$\varsigma(s) = 1 + \sum_{N=2} \frac{\boldsymbol{cos}[bLn(N)] \; - j\,\boldsymbol{sin}[bLn(N)]}{N^a} \tag{2.9}$$

The sum circular functions in (2.9) will be shown to diverge, and it is believed that they are possibly the slowest diverging series in any genuine mathematical problem/concept, i.e. if *b* is put to unity, in order to populate just one cycle requires a value of *N* increasing from 2 to ~2.22E156. This is due to the presence of the logarithm in the arguments.

For $\zeta(s)$ to have roots, the following identities must be simultaneously satisfied.

$$1 + \sum_{N=2} \frac{\boldsymbol{cos}[bLn(N)]}{N^a} = 0 \tag{2.10}$$

and

$$-\sum_{N=2} \frac{\boldsymbol{sin}[bLn(N)]}{N^a} = 0 \tag{2.11}$$

Now, as *N* increases, the number of values generated in each quadrant of these circular functions will, because of the presence of the logarithm in their argument, grow exponentially. This is illustrated in the following Table for the first three cycles with *b* set to 100.

Cycle	**1st Quadrant**	**2nd Quadrant**	**3rd Quadrant**	**4th Quadrant**	**Sum**
1	1	4	8	22	35
2	54	131	323	795	1303
3	1,955	4,809	11,827	29,090	47,681
Sum	2,010	4,944	12,158	29,907	49,019

Table 2.1 - Number of Values Generated in Each Quadrant of the First Three Cycles with *b* = 100.

Cosine is +ve in the 1st and 4th quadrants so that the sum Cosine Function continuously builds a surplus number of +ve values over the -ve. In the first three complete cycles this amounts to a surplus number of 14,815 +ve values, i.e. from the Table above

$$2{,}010 + 29{,}907 - 4{,}944 - 12{,}158 = 14{,}815 \quad (2.12)$$

Sine is +ve in the first two quadrants and -ve in the last two, so that the sum Sine Function builds up a surplus number of -ve values in the last two as follows

$$12{,}158 + 29{,}907 - 2{,}010 - 4{,}944 = 35{,}111 \quad (2.13)$$

All of these values will fall within the range -1 to +1.
Thus when (2.9) is calculated, with for example $a = 0$ and $b = 100$, both the Zeta Real Function, (2.10), and the Zeta Imaginary Function, (2.11), build up a diverging series of results as shown in the following Table of the first four cycles.

Results for *a* = 0, *b* = 100						
Cycles	Total Angle	Quadrant	No of Generated Values	N	Zeta Real	Zeta Imaginary
1	90	1	1	2	1.35	-0.94
	180	2	4	6	-1.68	-2.88
	270	3	8	14	-6.15	3.01
	360	4	**22**	**36**	**9.21**	**15.50**
2	450	1	54	90	39.09	-23.02
	540	2	131	221	-54.55	-95.79
	630	3	323	544	-234.64	134.23
	720	4	**795**	**1,339**	**331.37**	**577.48**
3	810	1	1,955	3,294	1,421.34	-814.54
	900	2	4,809	8,103	-2,002.81	-3,495.58
	990	3	11,827	19,930	-8,596.76	4,925.58
	1080	4	**29,090**	**49,020**	**12,115.31**	**21,144.98**
4	1170	1	71,551	120,571	52,009.23	-29,799.01
	1260	2	175,987	296,558	-73,293.24	-127,921.70
	1350	3	432,858	729,416	-314,635.65	180,272.95
	1440	4	**1,064,658**	**1,794,074**	**443,400.27**	**773,879.20**

Table 2.2 - Results for First Four Cycles with *a* = 0 and *b* = 100.

This Table illustrates that the Zeta Function possesses oscillatory characteristics that diverge in both real and imaginary directions with this setting. Results with a set to 0, 0.5, 1.0 and 1.5, all with b set to 100, are shown in Appendix A. In all of these results it is seen that the Riemann Zeta Function possesses similar oscillatory characteristics, and it is clear from (2.9) that b controls the frequency of the oscillations. Consequently, different values of b would not change the shape of the results, only their frequency related to N. Also if b is negative, exactly the same results would be obtained with the exception that the sum Sine results would change sign. On the other hand, a controls a damping effect so that its value determines whether the overall response diverges, oscillates or converges, with increasing N. In particular the following observations on the results in Appendix A are pertinent.

(i) $a < 0$. Results for this value of a are not shown in the Appendix, but it is obvious from (2.9), that they will rapidly diverge.

(ii) $a = 0$. Fig. (A.1), and Table 2.2 above, shows that the function diverges exponentially.

(iii) $a = 0.5$. Fig. (A.2) shows that the function still diverges exponentially, but at a slower rate than in (i).

(iv) $a = 1.0$ Fig.(A.3) shows that the oscillations are now close to stability. This is because the number of values generated in each quadrant is almost exactly proportional to *N*. Analysing to a depth of 15 decimal places shows that both real and imaginary parts are very slowly converging to values of 0.59 and -0.43 respectively.

(v) $a = 1.5$. Fig.(A.4). With this value of *a*, (and higher), the Zeta Function now converges very rapidly to constant non-zero real and imaginary values.

From (iv) above it is seen that there must be a value of *a* very slightly less than unity that would produce a stable oscillatory result.
It is also apparent that as *N* increases, many results, due to their oscillatory nature, pass through the zero point many times. However, due to their phase difference, the sum Cosine and sum Sine Functions do not pass through zero at the same value of *N*. This means that there is no partial sum for which the Zeta Function goes to zero.

From these results, and from (2.9), it is clear that there can be no combination of values of *a* and *b*, in particular with $a = 1/2$, that result in the identities of (2.10) and (2.11) being satisfied. Consequentially, the Riemann Zeta Function will have no non-trivial roots, and the Riemann Hypothesis is thereby determined to be false.

Because this result is contrary to current opinion, the analytical methods supporting that opinion are reviewed in the next section.

3.0 A Review of Current Procedures Used in the Study of the Riemann Hypothesis.

There are three main procedures central to the classical investigation of the Riemann Hypothesis. They are

(i) The Euler Product

$$\varsigma(s) = \prod_{N=p}\left(1 - N^{-s}\right)^{-1} \tag{3.1}$$

where *p* ranges over all the primes.

(ii) The relationship of the Zeta Function to the Eta Function.

$$\varsigma(s) = \frac{\eta(s)}{\left(1 - \frac{1}{2^{(s-1)}}\right)} \tag{3.2}$$

(iii) The functional equation of the Zeta Function.

$$\varsigma(1-s) = 2^{(1-s)}\pi^{-1}\boldsymbol{sin}\left(\frac{1-s}{2}\pi\right)(s-1)\boldsymbol{!}\,\varsigma(s) \tag{3.3}$$

Consider the first equation, the Euler Product. The correct method to generate this is as follows

$$\varsigma(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \cdots \tag{3.4}$$

To maintain term order integrity, now move all terms with even denominators to the LHS

$$\varsigma(s) - \frac{1}{2^s} - \frac{1}{4^s} - \frac{1}{6^s} - \cdots = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \cdots \tag{3.5}$$

which becomes

$$\varsigma(s)\left(1 - \frac{1}{2^s}\right) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \cdots \tag{3.6}$$

Now move all remaining terms with denominators divisible by three to the LHS, thus

$$\left(1 - \frac{1}{2^s}\right)\varsigma(s) - \frac{1}{3^s} - \frac{1}{9^s} - \frac{1}{15^s} = 1 + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{11^s} + \cdots \tag{3.7}$$

which becomes

$$\left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right)\varsigma(s) = 1 + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{11^s} + \cdots \tag{3.8}$$

and the process of continuously moving primes from the LHS to the RHS, one at a time, in this fashion eventually results in

$$\left(1 - \frac{1}{p_n^s}\right)\cdots\cdots\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{2^s}\right)\varsigma(s) = 1 + \frac{1}{p_{(n+1)}^s} + \cdots \tag{3.9}$$

Here it is classically stated that if this process is continued to a sufficient number of terms, the remaining term values on the RHS will approach zero, and (3.1) can be realised. This statement is not valid. Call the remaining terms $R(s)$, then

$$R(s) = \frac{1}{N_1^s} + \frac{1}{N_2^s} + \cdots \tag{3.10}$$

where the N_n^s are the remaining primes and their products, which are unlimited in number. By virtue of this, if s is complex, then, depending upon the value of a, $R(s)$ will either, along with the rest of the function, diverge to infinity, oscillate or converge to some finite number, according to the same process as in Section 2.0 above. If however, s is real and less than or equal to unity, then $R(s)$ will also diverge to infinity. Or if s is greater than unity, then $R(s)$ will converge to some positive number. As a result, (3.1) is in the last case, only an approximation, and in the rest, grossly in error. Accordingly, (3.1) as it is stated, is not valid and should not be used in analysis without the appropriate caveats.

The second equation identified above appears to be valid when $\eta(s)$ converges, because the transformation process can be effected with a single transition. Thus, maintaining term order integrity

$$\eta(s) = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \frac{1}{5^s} - \cdots \tag{3.11}$$

Now add twice the even terms to both sides thus

$$\eta(s) + 2\left(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \cdots\right) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \cdots \tag{3.12}$$

which becomes

$$\eta(s) + \frac{1}{2^{(s-1)}}\varsigma(s) = \varsigma(s) \tag{3.13}$$

so that

$$\eta(s) = \left(1 - \frac{1}{2^{(s-1)}}\right)\varsigma(s) \tag{3.14}$$

so that (3.2) is realised. However, if s is real and $0 < s < 1$, a convergent region of $\eta(s)$, (3.14) appears to state that, multiplying a divergent series by a finite number, results in a convergent series. This cannot be right. The correct interpretation should be that the above conversion process, is only valid when <u>both</u> series are convergent. Otherwise the relationship is not applicable.

Finally, consider the third process identified above as represented by (3.3). This relationship was, as shown in [2], derived from the Euler Product as represented by (3.1), and uses the relationship of (3.2) to effect the determination of $\zeta(1\text{-}s)$, so purportedly providing a method of deriving values of $\zeta(s)$ for both positive and negative s. However, in view of the dissertation above with regard to the Euler Product and the Eta Function relationship, the functional equation as represented by (3.3), is not considered valid. The overall result of this is that the combined use of the above three processes in determination of either trivial or non-trivial roots of the Zeta Function is also invalid.

The reason why the transformation process used above is "correct", is that it maintains term order. Rearranging terms of an infinite series can sometimes lead to erroneous results, depending upon the nature of the re-arrangement, and how the rearranged series is used. Because of the importance of this some examples are given in Appendix B.

4.0 Conclusions.

The result obtained here is most probably an unexpected one, bearing in mind the amount of work that has been conducted in the past to calculate the hundreds of millions of purported non-trivial roots of the Zeta Function, all with real part 1/2. All of these calculations were however, carried out using the processes discussed in Section 3.0 above, some of which were largely based upon the archive papers of Bernhard Riemann in Gottingen. It is thus considered that, unfortunately, the development of these methods, having been based upon a false premise concerning the mathematical treatment of infinite series, renders the above calculations invalid as roots of $\zeta(s)$. Consequently, it is suggested that they must therefore be the roots of some other infinite series, generated by the use of the three processes discussed in Section 3.0.

Accordingly, based upon the analysis in Section 2.0, it is concluded that the Zeta Function possesses neither trivial nor non-trivial roots. The hypothesis is consequently false.

The process used here, one of converting the imaginary part of the exponent to its circular function equivalents, may provide a useful way to determine divergence, convergence and other such characteristics of infinite series with complex exponents.

Finally, if the concerns regarding the mathematical treatment of infinite series, as highlighted here, are accepted and endorsed, it is suggested that the implications could be significant.

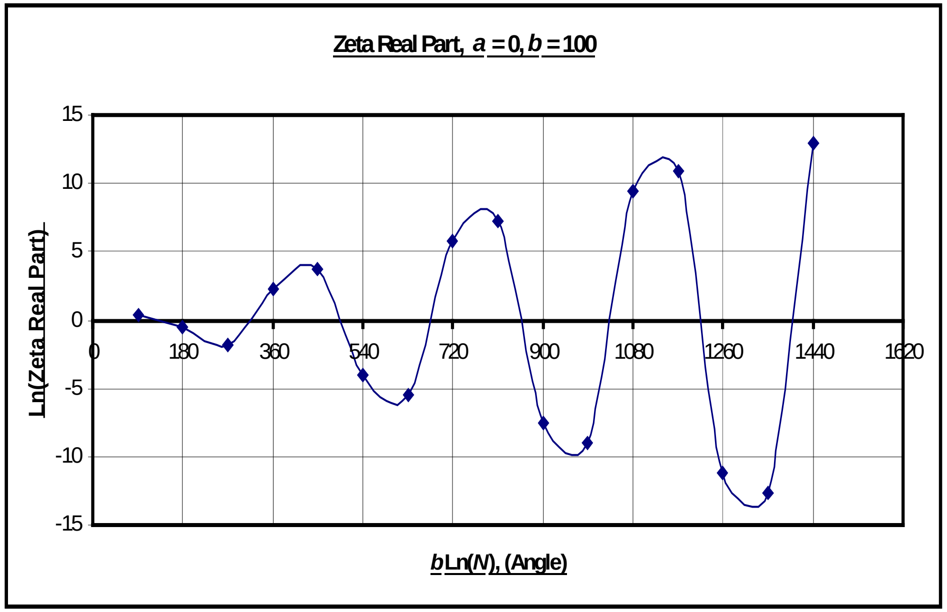
Appendix A.

Zeta Function Results

This Appendix provides results of the Zeta Function's characteristics generated from (2.9) via a macro driven EXCEL spreadsheet. They cover the range of *a* from 0 to 1.5 in steps of 0.5, for a value of *b* of 100. This value of *b* was chosen so that up to four cycles of results could be obtained in a reasonable time. Lower values of *b* would produce identical characteristics at a different oscillatory frequency, but would take an exponentially longer time.

Results for *a* = 0, *b* = 100						
Cycles	**Total Angle**	**Quadrant**	**No of Generated Values**	**N**	**Zeta Real**	**Zeta Imaginary**
1	90	1	1	2	1.35	-0.94
	180	2	4	6	-1.68	-2.88
	270	3	8	14	-6.15	3.01
	360	4	**22**	**36**	**9.21**	**15.50**
2	450	1	54	90	39.09	-23.02
	540	2	131	221	-54.55	-95.79
	630	3	323	544	-234.64	134.23
	720	4	**795**	**1,339**	**331.37**	**577.48**
3	810	1	1,955	3,294	1,421.34	-814.54
	900	2	4,809	8,103	-2,002.81	-3,495.58
	990	3	11,827	19,930	-8,596.76	4,925.58
	1080	4	**29,090**	**49,020**	**12,115.31**	**21,144.98**
4	1170	1	71,551	120,571	52,009.23	-29,799.01
	1260	2	175,987	296,558	-73,293.24	-127,921.70
	1350	3	432,858	729,416	-314,635.65	180,272.95
	1440	4	**1,064,658**	**1,794,074**	**443,400.27**	**773,879.20**

Fig.(A.1) - Zeta Function Characteristics for *a* = 0, *b* = 100.



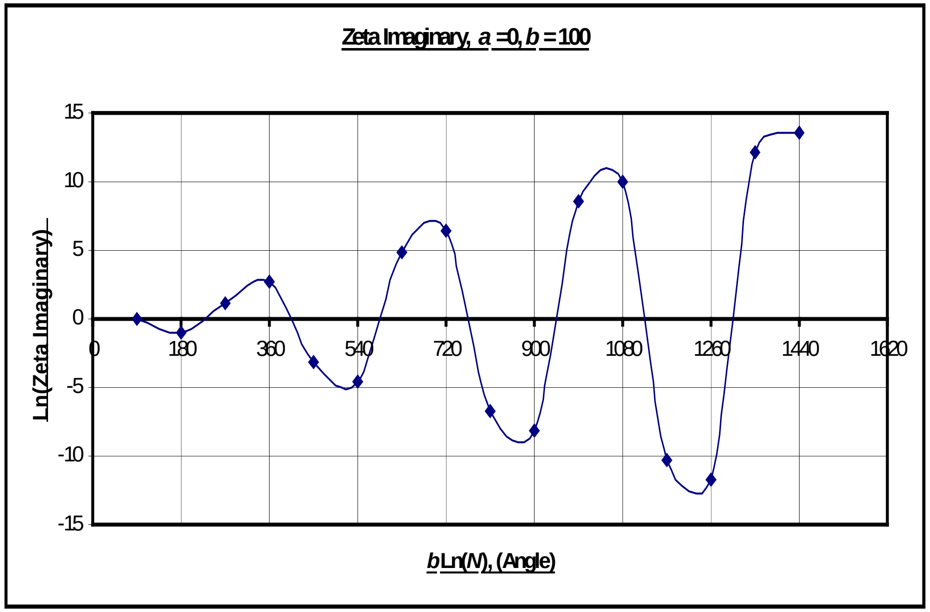
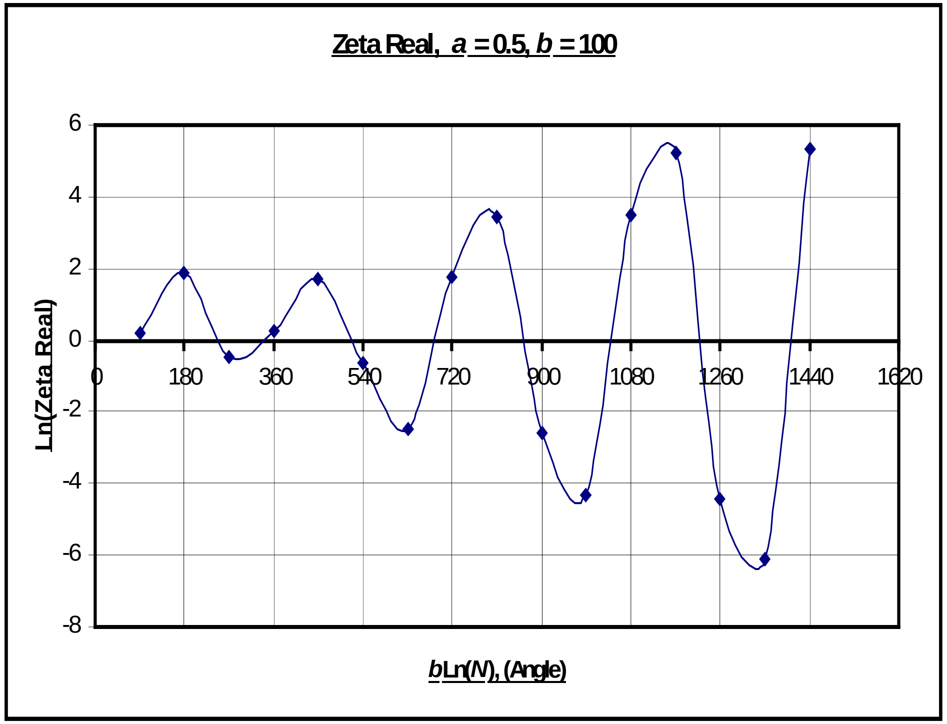


Fig.(A.1) - Zeta Function Characteristics for *a* = 0, *b* = 100.

The value axis is plotted to a *Ln* scale, so that a straight line joining the peaks indicates an exponential divergence.

Zeta Function Results for *a* = 0.5, *b* = 100						
Cycles	**Total Angle**	**Quadrant**	**No of Generated Values**	**N**	**Zeta Real**	**Zeta Imaginary**
1	90	1	1	2	1.25	-0.66
	180	2	4	6	-0.15	-1.69
	270	3	8	14	-1.64	0.09
	360	4	**22**	**36**	**1.30**	**2.81**
2	450	1	54	90	5.43	-1.88
	540	2	131	221	-1.86	-8.27
	630	3	323	544	-11.95	3.15
	720	4	**795**	**1,339**	**5.96**	**18.99**
3	810	1	1,955	3,294	30.80	-9.10
	900	2	4,809	8,103	-13.25	-48.06
	990	3	11,827	19,930	-74.34	21.02
	1080	4	**29,090**	**49,020**	**33.00**	**116.84**
4	1170	1	71,551	120,571	183.28	-53.07
	1260	2	175,987	296,558	-83.19	-288.75
	1350	3	432,858	729,416	-452.81	129.15
	1440	4	**1,064,658**	**1,794,074**	**202.59**	**708.83**

Fig.(A.2) - Zeta Function Characteristics for *a* = 0.5, *b* = 100.



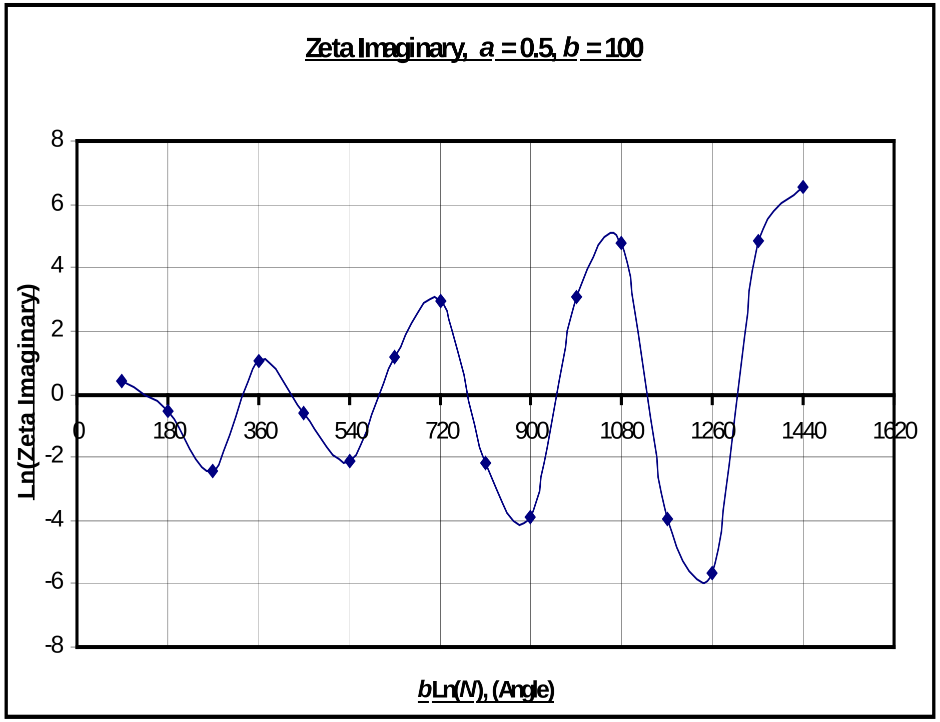
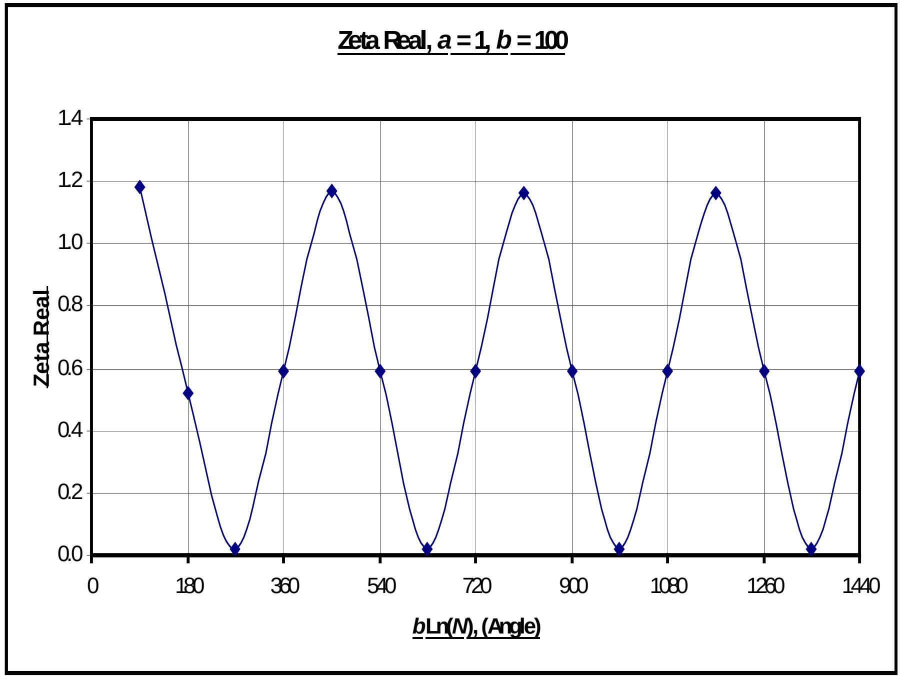
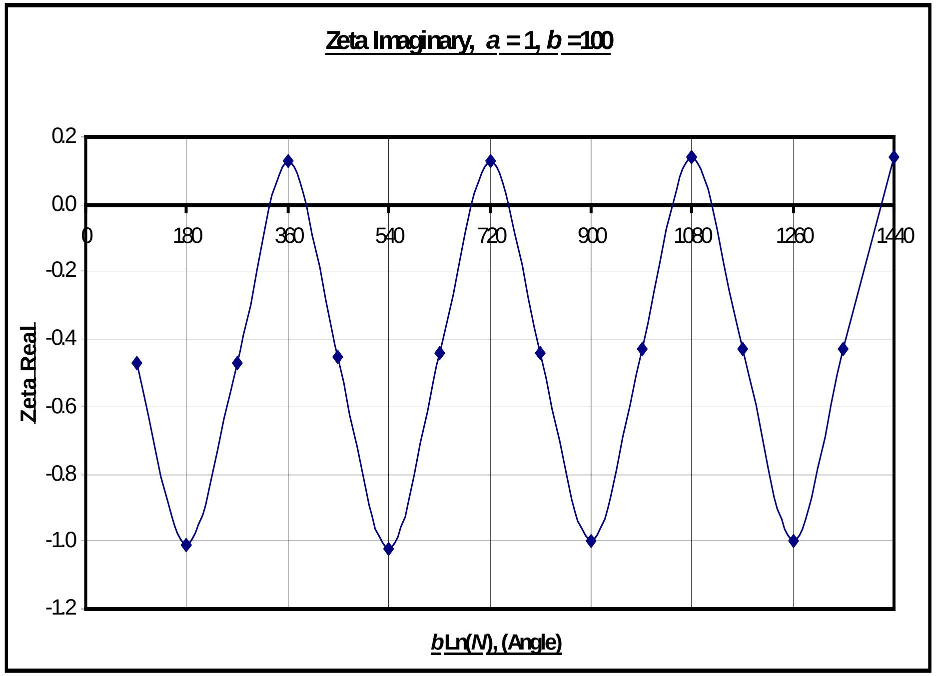


Fig.(A.2) - Zeta Function Characteristics for *a* = 0.5, *b* = 100.

Zeta Function Results for *a* = 1, *b* = 100						
Cycles	**Total Angle**	**Quadrant**	**No of Generated Values**	**N**	**Zeta Real**	**Zeta Imaginary**
1	90	1	1	2	1.18	-0.47
	180	2	4	6	0.52	-1.01
	270	3	8	14	0.02	-0.47
	360	4	**22**	**36**	**0.59**	**0.13**
2	450	1	54	90	1.17	-0.45
	540	2	131	221	0.59	-1.02
	630	3	323	544	0.02	-0.44
	720	4	**795**	**1,339**	**0.59**	**0.13**
3	810	1	1,955	3,294	1.16	-0.44
	900	2	4,809	8,103	0.59	-1.00
	990	3	11,827	19,930	0.02	-0.43
	1080	4	**29,090**	**49,020**	**0.59**	**0.14**
4	1170	1	71,551	120,571	1.16	-0.43
	1260	2	175,987	296,558	0.59	-1.00
	1350	3	432,858	729,416	0.02	-0.43
	1440	4	**1,064,658**	**1,794,074**	**0.59**	**0.14**

Fig.(A.3) - Zeta Function Characteristics for *a* = 1.0, *b* = 100.

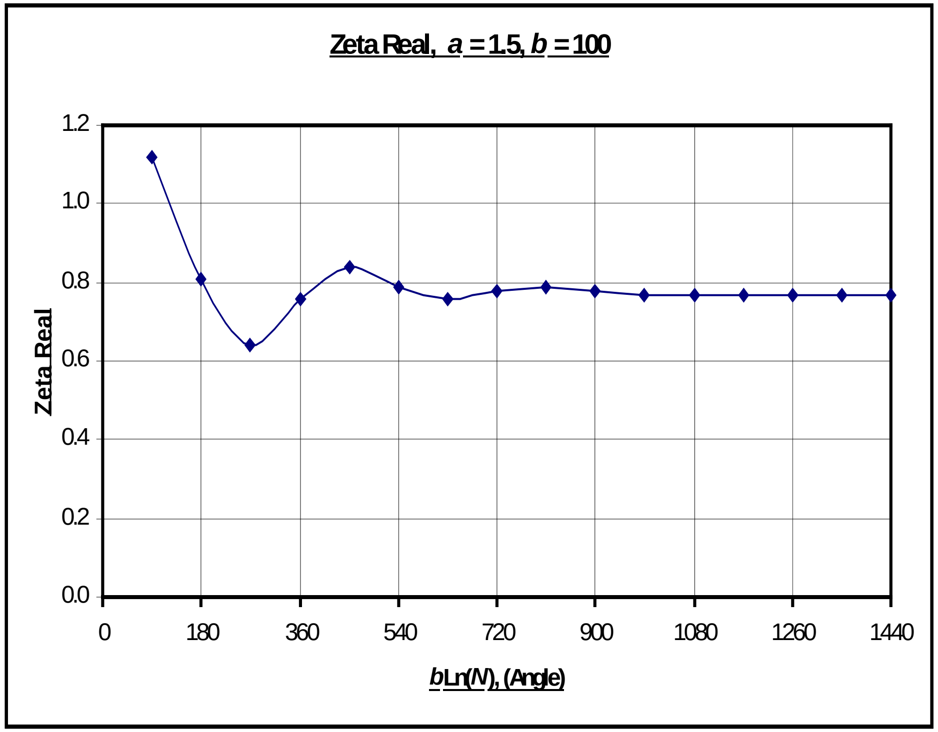




(A.3) - Zeta Function Characteristics for *a* = 1.0, *b* = 100.

Zeta Function Results for *a* = 1.5, *b* = 100						
Cycles	**Total Angle**	**Quadrant**	**No of Generated Values**	**N**	**Zeta Real**	**Zeta Imaginary**
1	90	1	1	2	1.12	-0.33
	180	2	4	6	0.81	-0.62
	270	3	8	14	0.64	-0.46
	360	4	**22**	**36**	**0.76**	**-0.32**
2	450	1	54	90	0.84	-0.40
	540	2	131	221	0.79	-0.45
	630	3	323	544	0.76	-0.42
	720	4	**795**	**1,339**	**0.78**	**-0.40**
3	810	1	1,955	3,294	0.79	-0.41
	900	2	4,809	8,103	0.78	-0.42
	990	3	11,827	19,930	0.77	-0.42
	1080	4	**29,090**	**49,020**	**0.77**	**-0.42**
4	1170	1	71,551	120,571	0.77	-0.42
	1260	2	175,987	296,558	0.77	-0.42
	1350	3	432,858	729,416	0.77	-0.42
	1440	4	**1,064,658**	**1,794,074**	**0.77**	**-0.42**

Fig.(A.4) - Zeta Function Characteristics for *a* = 1.5, *b* = 100.



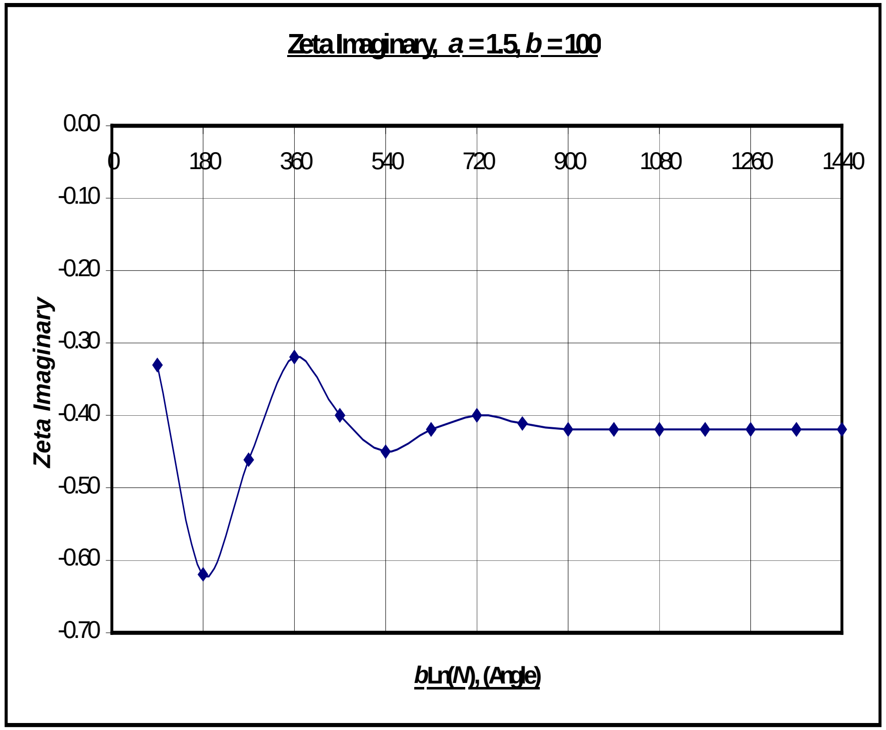


Fig.(A.4) - Zeta Function Characteristics for *a* = 1.5, *b* = 100.

Appendix B.

The Importance of Term Order in Infinite Series.

The importance of term order in infinite series is demonstrated here by way of four examples. The first, drawn from [1], is as follows.

It is well known, and easily derivable from Newton's expansion of *Ln*(1-*x*), that

$$Ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots \tag{B.1}$$

(Note that this is the Eta Function with $s = 1$). Re-arranging the series as follows

$$Ln(2) = 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \cdots \tag{B.2}$$

and putting in some parentheses

$$Ln(2) = \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \cdots \tag{B.3}$$

and making the arithmetical calculations within the parentheses gives

$$Ln(2) = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \cdots \tag{B.4}$$

To finally give

$$Ln(2) = \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots\right) \tag{B.5}$$

Which seems to suggest that

$$Ln(2) = \frac{1}{2} Ln(2) \tag{B.6}$$

This is of course nonsense, and the problem lies in the re-arrangement at (B.2) to (B.5). If the partial sums of series (B.1) and (B.2) are graphed, they are shown in Fig. B.1 below.

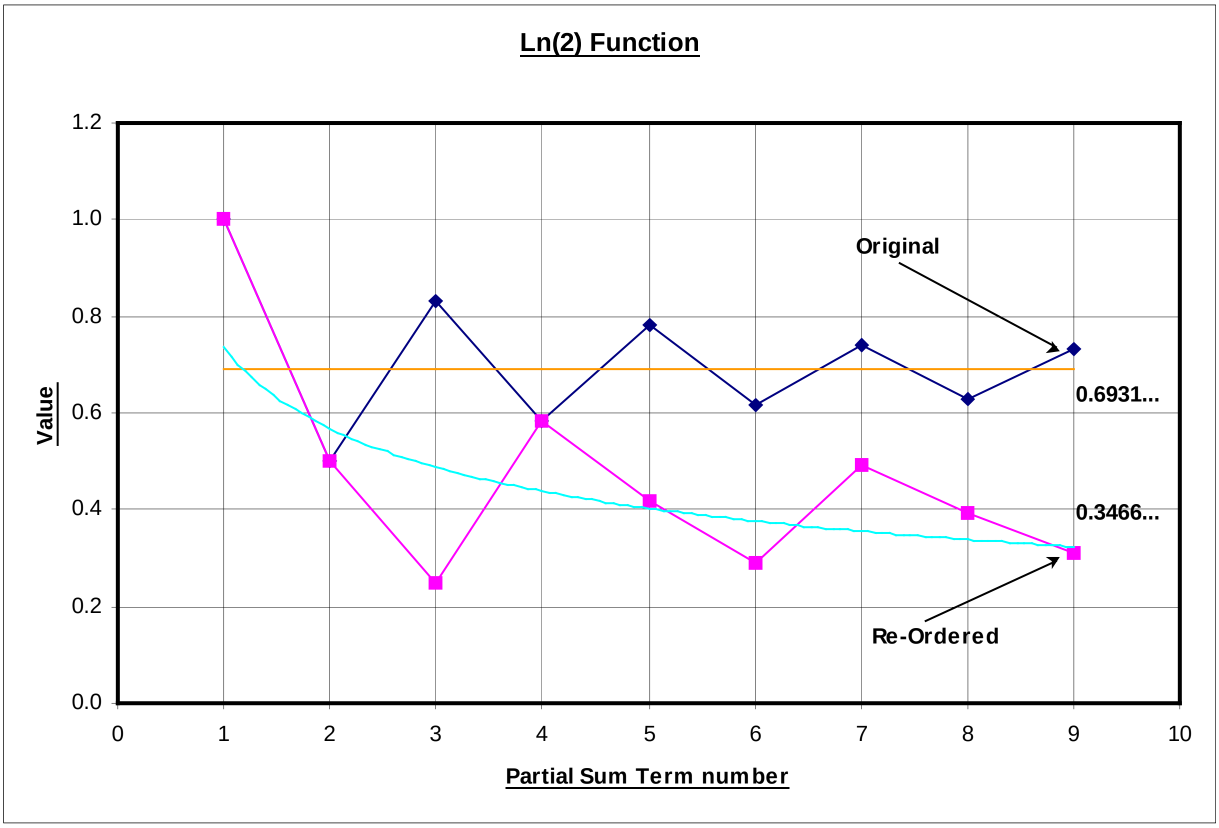


Fig. B.1- Partial Sum Sequence of Ln(2), Original and Re-Arranged

Fig. B.1 clearly shows that (B.1) and (B.2) are distinctly different infinite series. They converge to different sums and their convergence paths are unrelated. Calling series (B.1) "conditionally convergent" does not alleviate the problem. Note that graphing (B.4) or (B.5) produces yet a different series.

In the second example, consider again the Eta Function with s = 2.

$$\eta(2) = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \cdots \tag{B.7}$$

Now re-order the terms to obtain

$$\eta(2) = 1 + \frac{1}{9} - \frac{1}{4} + \frac{1}{25} - \frac{1}{16} - \cdots \tag{B.8}$$

Graphing both series gives

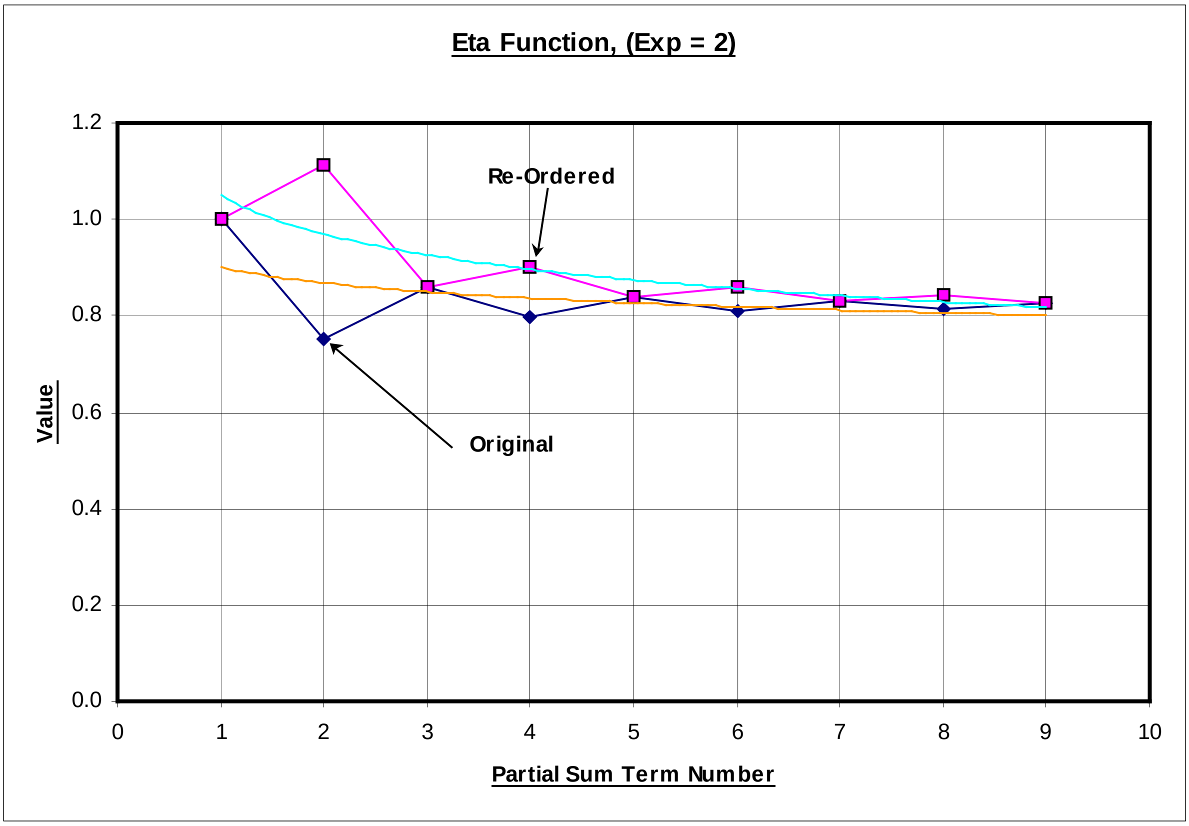


Fig. B.2 - Partial Sum Sequence of η(2), Original and Re-Arranged.

Once again these series are clearly different. While they converge to the same sum, their convergence paths are different.

A second, more extreme re-arrangement of the Eta Function is achieved by placing all the negative terms before the positive, thus

$$\eta(2)=1-\frac{1}{4}-\frac{1}{16}-\frac{1}{36}-\cdots+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\cdots \tag{B.9}$$

Graphing (B.7) and (B.9) gives

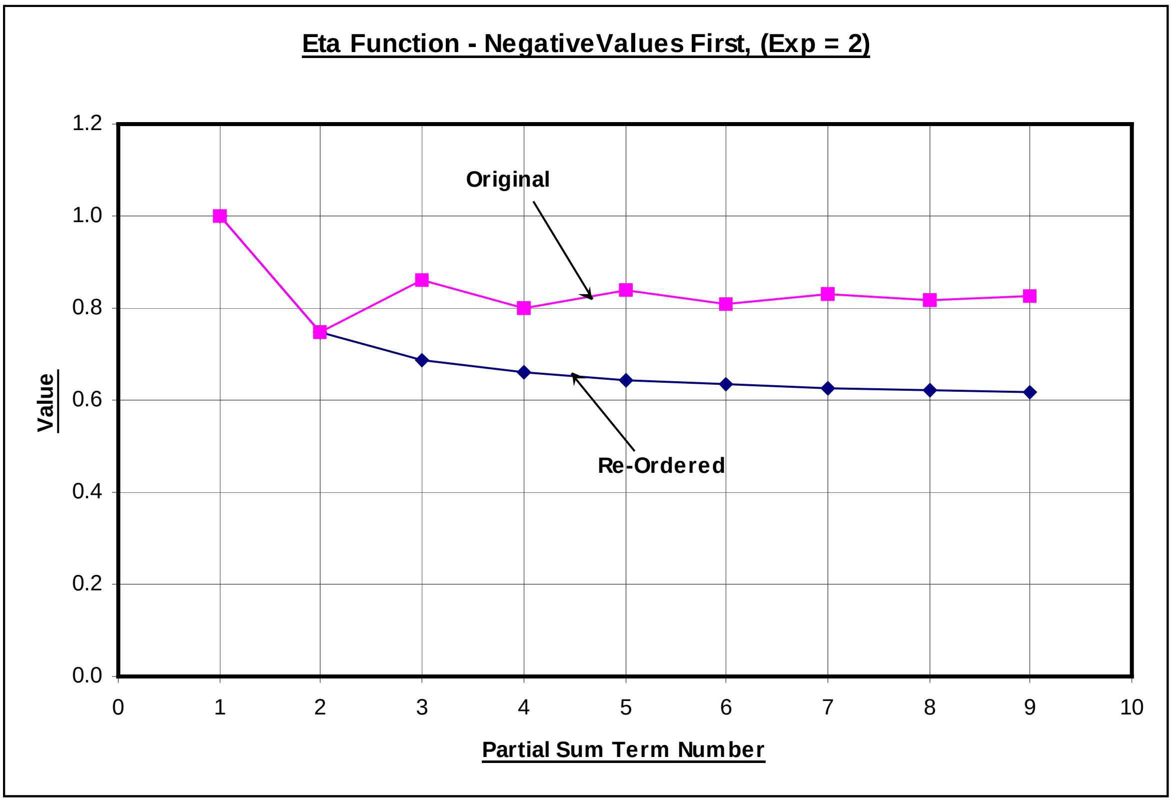


Fig. B.3 - Partial Sum Sequence of η(2), Original and Extreme Re-Arrangement.

This shows that the two series have become widely separated. They will eventually converge to the same sum, but only after the negative terms have converged to a final sum after an unlimited number of terms, then followed by the unlimited number of positive terms. It is perhaps not clear how this should be interpreted.

Finally, consider the Harmonic series, a relative of the Zeta Function.

$$\varsigma(1) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots \tag{B.10}$$

The simple re-arrangement of exchanging terms in pairs yields

$$\varsigma(1) = 1 + \frac{1}{3} + \frac{1}{2} + \frac{1}{5} + \frac{1}{4} + \cdots \tag{B.11}$$

and graphing gives

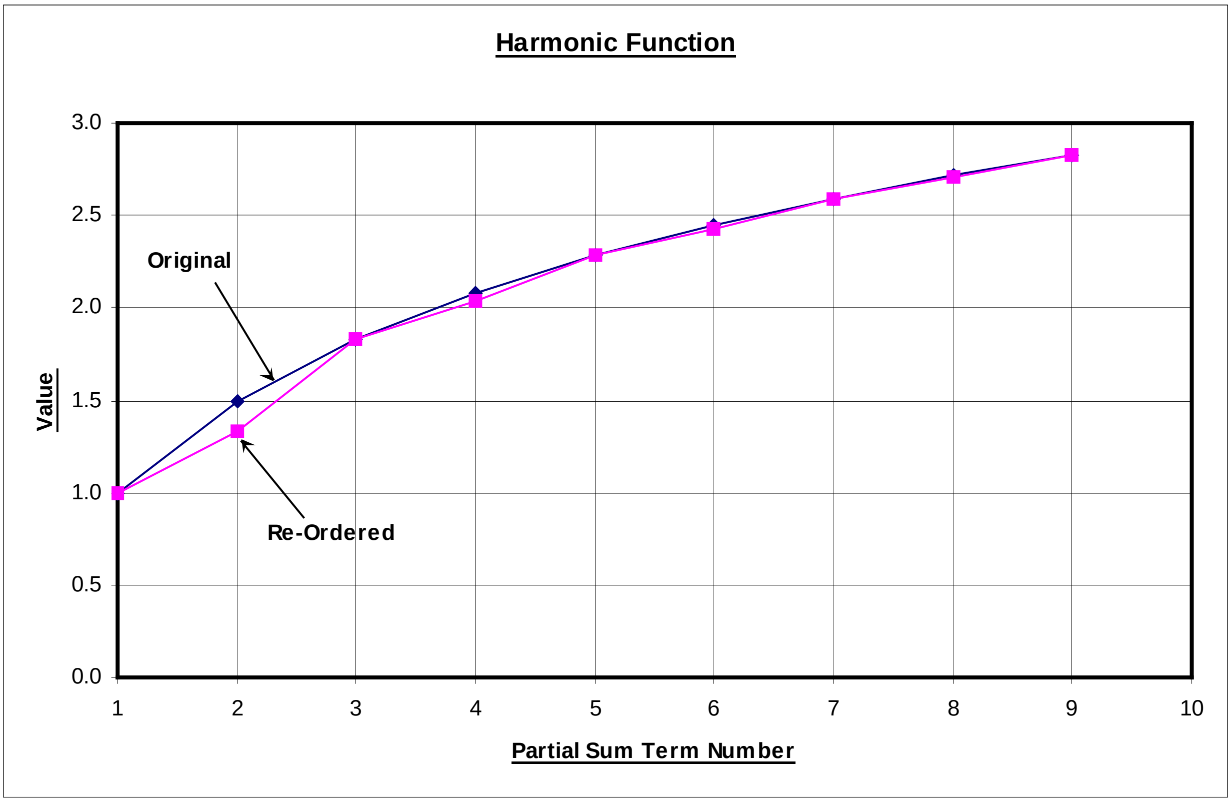


Fig. B.4 - Partial Sum Sequence of the Harmonic Series Original and Re-Arranged.

Here the problem is less acute. While the two series are different, they follow similar divergent paths and the partial sum terms are closer than in the above examples.

These four examples have clearly demonstrated that changing the term sequence in order to manipulate series into forms for specific calculations can give erroneous results. As shown, in some cases, where only positive terms are present, and the term order variations are not extreme, the errors may be small. Where negative terms are involved, and/or term order variation is extreme, the errors can be gross.

This demonstrates that the veracity of any infinite series is entirely dependent upon the order of its terms.

References.

[1] John Derbyshire, *Prime Obsession*, Plume, Penguin Group, 2003.

[2] H.M.Edwards, *Riemann's Zeta Function*, Academic Press, 1974, (Dover Edition, 2001).