GENERATION OF PRIME NUMBERS

BY SINGLE VARIABLE POLYNOMIAL

EQUATIONS.

2

THE METHODS OF COMPOSITE SEARCH

AND ULAM'S SPIRAL, (EXTENDED).

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ABSTRACT.

This paper continues the process of generating prime number sequences, using second order quadratics, by describing two semi-analytic methods, Composite Search and a re-configurable Ulam's Spiral.

1 Introduction.

In the first paper on this subject, the small total of seven quadratics were shown to generate sequences of prime numbers covering all those between 0 and 100 plus 57 others. Of these, five, including Euler's and Legendre's, were already well known and documented.

Using the methods described below, the list has now been expanded to 451 quadratics of which 104 cover all the prime numbers between 0 and 1500 plus 190 beyond. The profusion of acceptable equations is now such that it is possible to tighten the control criteria as will be described below.

It is hoped that as more quadratics meeting the criteria are found, a pattern will emerge that will enable others to be "predicted" according to some "pattern law". However, it is clear that for the first term in any sequence to be prime, the quadratic constant coefficient, a_0, in the generating equation must be prime. In addition, for the second term in the sequence to be prime, the sum of all the coefficients must also be prime. Because the distribution of prime numbers is non-linear, so then will be quadratics that generate sequences of primes, according to the criteria, when based upon these two factors. Consequently, any pattern law that does subsequently emerge, can only be based upon the first two coefficients, a_2, and a_1 alone.

In this paper it will be seen that the number one, (#1), is treated as a prime number. This is contrary to classical opinion. However, it conforms to the definition of a prime number in that it can only be divided by unity and itself. The fact that "unity" and "itself" are one and the same, and the same as the number itself, is not considered sufficient reason to exclude it from the class of primes.

2   Prime Number Sequence Generation.

2.1   Preamble, Amendment of the Control Criteria and Further Categorisation.

In [1], quadratics generating prime number sequences were categorised into four groups:

(i) Full Prime - Equations that generated a sequence of primes starting with a_0 and of at least $(a_0 - 1)$ in number, (minimum 4).

(ii) Sub-Prime - Equations that generated a sequence of primes starting with a_0 and of at least 4 in number.

(iii) Symmetrical V-Prime - Equations that generate two sequences of primes, the first descending from a maximum to a minimum term, and the second increasing from the same minimum to the same maximum.

(iv) Unsymmetrical V-Prime - Equations similar to (iii) but generating arms of different lengths.

In this paper the last two types are not further considered because they are all convertible to either of the first two types by the reduction process described in [1] and this paper.
The number of equations now produced, both Full Prime, 357, and Sub-Prime, 94, is such that the control criteria can be tightened. Thus the minimum permissible is now increased from four to five. This change reduces the number of acceptable quadratics found to 222, Full Prime, and 59, Sub-Prime, within which the 104 generating all the primes from 0 to 1500 plus 190 others is contained. There is only one exception to this change which must be accepted. There is only one equation that generates the prime number 3 in any acceptable sequence. That quadratic is

$$N = x^2 + x + 1 \tag{2.1}$$

Eq.(2.1) generates 1, 3, 7 and 13 for $x = 0$ to 3 and fails at $N = 21$ for $x = 4$. (Equations with $a_0 = 3$ can only generate a sequence of 3 primes).
The remaining Full and Sub-Prime quadratics making up the compliment of acceptable equations can now be further categorised into three sub-groups as follows.

(i) Type (a) - Equations of the form:

$$N = a_2 x^2 + a_0 \tag{2.2}$$

Full Prime equations of this class generate a continuous sequence of primes, the quantity of which is equal to the value of a_0. Legendre's equation is one such.

(ii) Type (b) - Equations of the form:

$$N = a_2 x^2 + a_2 x + a_0 \tag{2.3}$$

Full Prime equations of this class generate a continuous sequence of primes, the quantity of which is equal to the value of $(a_0 - 1)$, except when $a_2 = 6$, when the quantity is equal to $(a_2 - 2)$. Euler's equation is of this type.

(iii) Type (c) - Equations of the form:

$$N = a_2 x^2 + a_1 x + a_0 \tag{2.4}$$

Full Prime equations of this class generate a continuous sequence of primes, the quantity of which is equal to the value of a_0.

Note that in the above three classes, when $a_0 = 1$, the number of primes generated must be equal to or greater than five.

It is not possible to list all the equations found in the text of this paper, and so they are contained, together with all their characteristics, in one of the spreadsheets associated with the paper, and described in Section 3.0.

2.2   The Composite Search Method.

The manner in which this method works is to evaluate whether any of the composite numbers lying within the range of the quadratic under evaluation, are generated by that quadratic. Consider the full prime equation of (2.2). Assume that this quadratic generates a composite number when $x = a_0 - n$. Substituting in (2.2)

$$N = a_2 \left(a_0 - n \right)^2 + a_0 \tag{2.5}$$

Expanding and re-arranging gives

$$\frac{N - \left(a_2 a_0 + 1 \right) \, a_0}{a_2} = n^2 - 2na_0 \tag{2.6}$$

Completing the square and re-arranging for n gives

$$n = - \left(\frac{N - a_0}{a_2} \right)^{1/2} + a_0 \tag{2.7}$$

So that if n is less than or equal to the set criteria, the quadratic under evaluation is acceptable. Equivalent algorithms for Type (b) and Type (c) equations are as follows.

For Type (b)

$$n = - \left(\frac{N - a_0 + a_2/4}{a_2} \right)^{1/2} + a_0 + \frac{1}{2} \tag{2.8}$$

and for Type (c)

$$n = - \left(\frac{N - a_0}{a_2} + \frac{a_1^2}{4a_2^2} \right)^{1/2} + a_0 + \frac{a_1}{2a_2} \tag{2.9}$$

In the above three algorithms, by replacing N by pq, where p and q are odd integers generating all the composites within the applicable range of the quadratic under evaluation, n can be evaluated. When n is integer, the quadratic under evaluation then generates the composite represented by the product pq. The equation then passes or fails according to whether n meets the criteria associated with (2.2) to (2.4) above.

2.3   The Re-Configurable Ulam's Spiral Method.

Ulam's Spiral, discovered by Stanislav Ulam in 1963, is a visual means of identifying quadratic equations that generate prime number sequences, and is very well documented in the mathematical literature. In its basic form it generates quadratics of the form

$$4x^2 + a_1 x + a_0 \tag{2.10}$$

The value of a_0 is that number at the heart of the spiral, (the Base Number), while the value of a_1 depends upon the direction in which the terms of the equation line up, viz, Fig. 2.1 below.
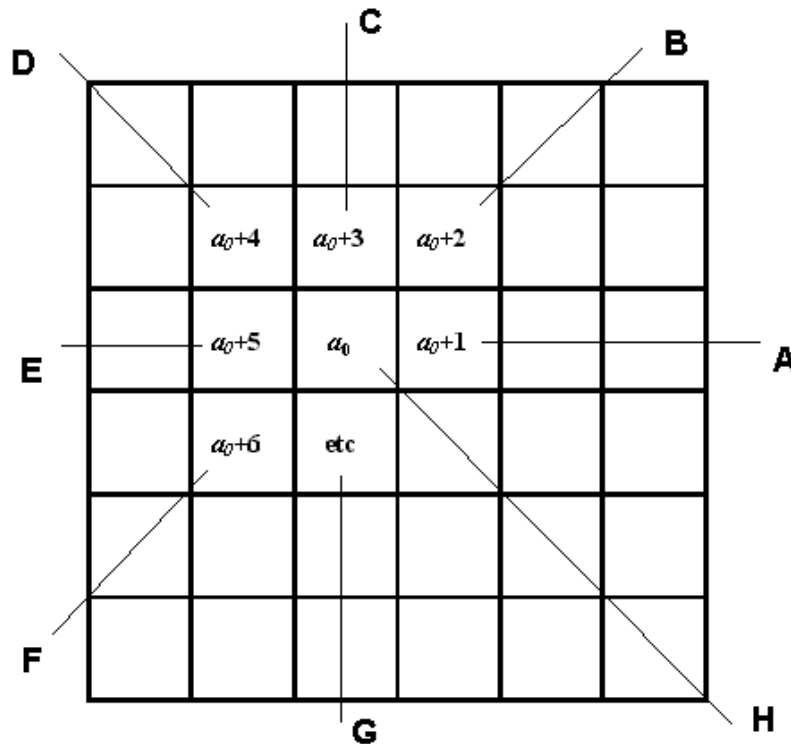

Fig. 2.1 - The Basic Ulam Spiral.

The value of a_1 in (2.10), according to the direction A to H, is shown in the following table.

Direction	a_1
A	- 3
B	- 2
C	- 1
D	0
E	1
F	2
G	3
H	4

Table 2.1 - Value of a_1 in Ulam's Spiral.

Thus it is clear from the above that, the terms of Type (a) equations always lie in direction D, and those of Type (b) in direction H. All other directions produce Type (c) equations.

Now, the characteristics of (2.10) and Table 2.1 only apply when the Step Size in the cells of the spiral is unity, as shown in Fig. 2.1. For the spiral to show the terms of an acceptable quadratic in which a_2 differs from the value 4, it is necesary to vary this Step Size. For instance, if, in Fig. 2.1, the value of a_0 is made equal to 41, then Euler's equation, Type (b), lines up along directions B and F as alternating terms either side of the Base Number. Changing the Step Size to 0.25, and re-configuring the spiral, then lines up the terms of Euler's equation along direction H. Thus the Step Size can be made any size desired, integer or fractional, which will subsequently allow for the identification of quadratics with values of a_2 and a_1 different from those in (2.10) and Table 2.1. Also, it is clear from the above that to portray a Full Prime equation on the spiral, the first term must appear as the Base Number, (except when a_0 is unity). If the first term appears in any other location, a Sub-Prime quadratic is produced. Another very important feature of the spiral is that it is not only the above eight directions of terms that produce quadratic equations, directions as shown in the following figure do also.
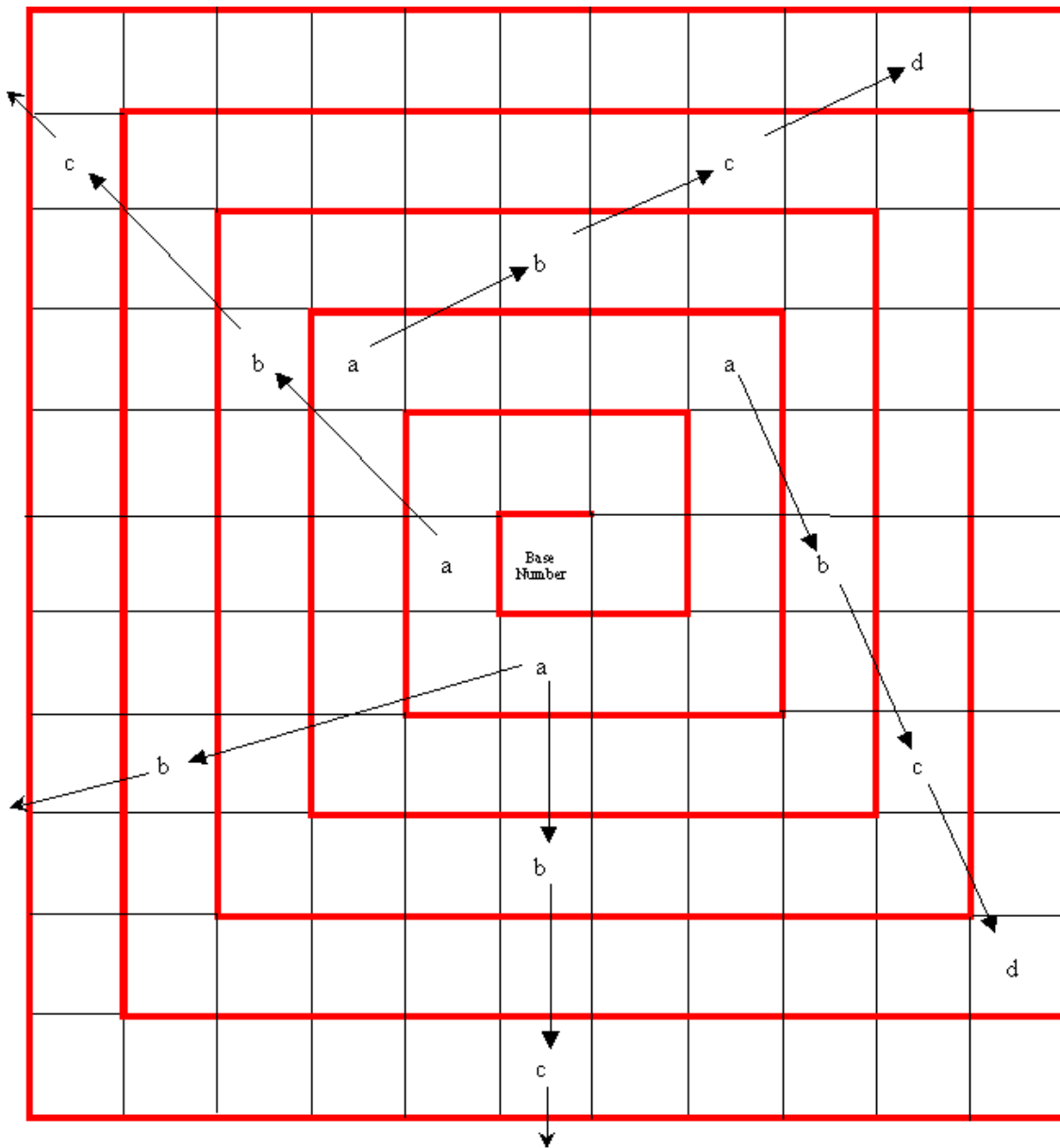
Fig. 2.2 - Acceptable Term Spacing in Ulam's Spiral.

There are essentially only two specific requirements for the terms in Ulam's Spiral to generate a quadratic, (prime generating or otherwise), They are

(i) The term values must be such that in Fig. 2.2, $a < b < c < d$.

(ii) Each term <u>must </u>cross at least one spiral boundary, and the number of boundaries crossed for each term must be the same. These boundaries are highlighted in Fig. 2.2,

An analytical evaluation of Ulam's Spiral, as extended in this paper, is presented in Appendix A.

3   Computerisation.

The above two methods of identification of prime number generating quadratics are computerised as macro drive EXCEL spreadsheets that can be downloaded as a ZIP file here:

www.relativitydomains.com/mathematics/primes2/PrimeGen.zip.

This file contains two spreadsheets, PrimeGen.xls and Ulam.xls. For better appreciation of the remainder of this paper, these files should be used in conjunction with this Section as required, but this is not essential.

3.1   The Composite Search Method.

The main sheet of PrimeGen.xls is shown in the figure below

Fig. 3.1 – PrimeGen.xls Main Sheet.

The use of this spreadsheet is described as follows. To test quadratics for full primeness, there are two modes.

(i) <u>Search Mode</u> - Depress the "Search Type" button to reveal "Full Prime". Depress the "Full Prime Mode" button to reveal "Auto". Insert the Coefficients of a quadratic into the "Coefficient Input" cells, ensuring that a_0 is a prime number and that the sum of the coefficients is also prime. Press the "START" button. The quadratic will then be tested for full primeness and if it passes the process will stop and an appropriate message will be displayed. The prime numbers generated can then be read off under "Numbers Generated".

If the quadratic fails, the appropriate coefficients, according to the quadratic Type, will be automatically increased by 2 and the test restarted. This process will be repeated until a Full Prime equation is found or until the ESC key is pressed on the keyboard.

(ii) <u>Single Test Mode</u> - To test an individual quadratic, set "Search Type" to "Full Prime" and "Full Prime Mode" to "Single". Insert the coefficients and press "START". The quadratic will then be tested and will pass or fail with the appropriate message being displayed and the process terminating.

To test quadratics for sub-primeness, there is only one mode as follows.

(i) <u>Sub-Prime Test</u> - Set "Search Type" to "Sub-Prime", the "Full Prime Mode" will automatically be set to "Single". Insert the coefficients and press "START". The quadratic will then be tested. The process will terminate when, within the test range, the lowest composite generated by the quadratic is found. A message will then be displayed showing the number of primes generated lower than the above composite, together with the appropriate values of the independent variable. The former can then be read off under "Numbers Generated". If a composite is not found within the test range, the process will terminate, a message displayed accordingly, and the test can be restarted with a higher test range.

In the above two processes, in some cases, some of the numbers shown under "Numbers Generated" will be too large for the applicable cell width. Pressing "the "Adjust Column Width" button will reveal all numbers. Pressing this button again, reverses the change.

There are two further points to note. First, testing quadratics with very large coefficients, especially a_2 and a_0, will take a considerable amount of time. It may therefore be preferable to simply enter the quadratic coefficients, and check the numbers generated against a table of primes. Such a table can be generated with the Visual Basic program, "Primes.exe", which may be obtained here.

www.relativitydomains.com/mathematics/primes/Primes.exe

The feature at the bottom of the sheet, "Equation Reduction Calculator", is mainly for use in conjunction with Ulam.xls and is described in the next Section.

The second sheet of this spreadsheet, presents a summary of all the acceptable quadratics found to date, their primary characteristics, and the prime numbers they generate. This covers all prime numbers between 0 and 1500 plus the 190 others. There are of course many duplications, but those quadratics generating a prime for the first time, are highlighted as "Unique" together with the prime number concerned. This feature has not been optimised so it may be possible to reduce the number of "Unique" quadratics whilst still covering the above range. A filter has also been activated to allow inspection of any particular feature of any equation. The area below the listing is available for input of additional equations.

3.2   The Re-Configurable Ulam Spiral Method.

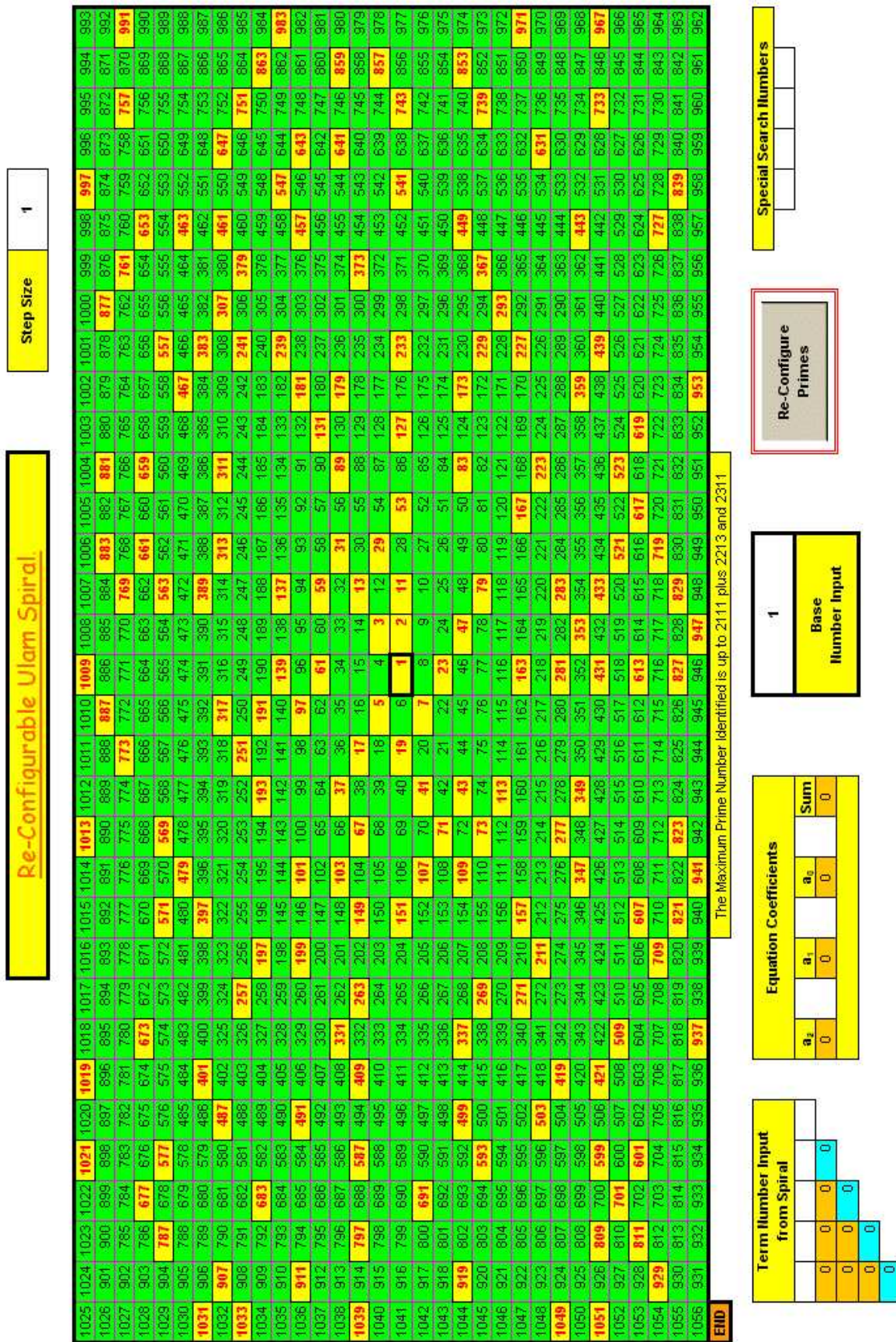The main sheet of Ulam.xls is shown below.

Fig. 3.2 – Ulam.xls Main Sheet.

In Fig. 3.2 the highlighted red numbers are primes, the rest composites. Data inputs that affect the spiral directly are, (i) "Base Number Input" which will be mostly integers, both positive and negative, but fractional numbers can also be input. (ii) "Step Size", this must be positive but can also be fractional as well as integer. (iii) "Special Search Numbers", up to four prime numbers can be entered here.

After changing any of the above three inputs, the "Re-Configure Primes" button should be pressed. This will completely reconfigure the entire spiral to the parameter inputs. All prime numbers will be highlighted as in Fig. 3.2 above, in their new positions. Those primes searched for in (iii) will be differently highlighted to enable easy location. When four or more primes line up in any of the eight major directions, or as in any of the paths shown in Fig. 2.2, anywhere on the spiral, the first four, or five terms of the sequence can be inserted into the "Term Number Input from Spiral" cells, and if they possess acceptable quadratic form, the coefficients of the applicable equation will be displayed in the "Equation Coefficients" cells. Otherwise an error message will be displayed. The quadratic can then be maximised for the number of primes generated, as described below.

3.2.1   Maximisation of the Sequence.

Once an acceptable quadratic has been found in the spiral, its coefficients should be entered into the "Equation to be Reduced" cells on PrimeGen.xls. In the cell marked "P", prime numbers lower than a_0 should be tried until the smallest one is found that produces integer values in the N+ and N- cells. These are the coefficients of the reduced equation(s). Their sum must be a prime number. If so, they should then be entered into the "Coefficient Input" cells at the top of the sheet and the numbers generated checked for three features.

 (i) The reduced equation generates the original four primes.

 (ii) No composites appear in the sequence lower than these original four primes.

 (iii) At least a sequence of five primes are generated.

If these conditions are satisfied, the quadratic sequence has been maximised. If they are not satisfied, further values of "P" should be tried until they are, or the original quadratic is seen to be already maximised.

Once the above process has been completed, it may be desired to display all the generated primes in line on the spiral. Two methods to do this are demonstrated in Appendix B.

3.3   Finding Equations for Difficult Numbers.

When the above two processes fail to find quadratics that generate particular prime numbers, the following manual approach can be adopted.

In Ulam.xls insert the problem number, P_3, into the third cell from the left in the cells labelled "Term Number Input from Spiral". Now in the first and second cells, insert two primes, P_1 and P_2, ($P_1 < P_2$), such that the numbers that appear in the immediate cells below, (a and b), are positive, thus

Term Number Input from Spiral				
P_1	P_2	P_3		
a	b	$-P_3$	0	
c	-d	P_3		
e	f			
-g				

Fig 3.3 - Finding Equations for Difficult Numbers, (i).

If the number "e" is prime, insert it as P_4 into the cell to the right of P_3 to give the configuration in Fig. 3.4 below. If "e" is not prime, vary P_2 up or down to the next prime until "e" does become prime.

Term Number Input from Spiral				
P_1	P_2	P_3	P_4	
a	b	h	$-P_4$	
c	c	-i		
0	-g			
g				

Fig. 3.4 - Finding Equations for Difficult Numbers, (ii).

At this point an equation will appear in the "Equation Coefficients" cells. This is the quadratic of the sequence, and if "g" is prime, the quadratic generates at least five primes including the difficult number P_3. The quadratic can now be maximised as in the process above. If "g" is not prime, repeat the above procedure, varying P_2 until both "e" and "g" become prime.

An example of a difficult number is 467. Using the above procedure, with $P_1 = 1$, $P_2 = 223$, and $P_3 = 467$, P_4 becomes 733 and "g" 1021. The quadratic is

$$N = 11x^2 + 211x + 1 \tag{3.1}$$

The second sheet in Ulam.xls, is a simple macro driven feature to check large integers for factors.

4   Conclusions.

The two methods described in this paper have produced quadratics that generate all the 240 prime numbers between 0 and 1500. By fully exploring the flexibility of the extended Ulam's Spiral, and performing extended searches with the Composite Search method, it is believed that many more could be found.

The highest prime number generated to date by a quadratic found in this exercise is 15,319. To extend significantly beyond this would require a much faster computing capability and a much larger Ulam Spiral, probably one that would have to be scrolled. Also, coding in a high level language, to produce a stand alone EXE file would be essential for very large numbers to be handled.

Finally, in the next paper on this subject, the first pattern in these quadratics will be described.

This paternation will allow their prediction, rather than the necessity for searching, and will establish sets of "families" of quadratics, which will permit the generation of a possibly limitless number of prime number sequences extending into very high values.

APPENDIX A.

An Analytical Evaluation of Ulam's Spiral.

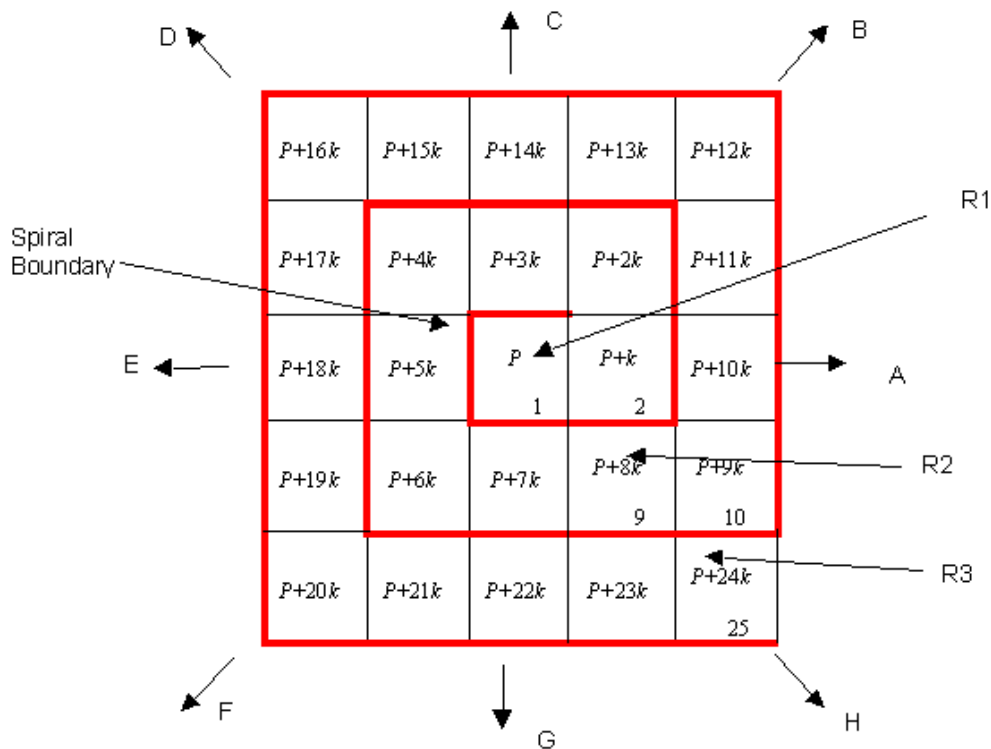An analytical representation is shown in Fig.A.1 below.

Fig. A.1 - Ulam's Spiral, Analytical Representation.

In Fig. A.1 the following parameters are defined as follows:-

  (i) P is the Base Number in the Base Cell.

 (ii) $P+nk$ are the values in all other Cells.

(iii) k is the Step Size.

 (iv) The cells are numbered 1, the Base Cell, to 25 etc around the spiral.

  (v) R_n is the Spiral Ring number, R_1 contains just the Base Cell, ($R_1 = 0$),

       R_2 contains cells 2 to 9, ($R_2 = 1$), R_3 contains cells 10 to 25, ($R_3 = 2$) etc.

 (vi) D is the direction value in which successive terms of a quadratic lie. $D = $ - 3 to + 4 for directions A to H, (as in Table 2.1).

(vii) B is the number of spiral boundaries crossed by successive quadratic terms.

(viii) n_r is the number of cells counting from the first cell in a ring, to the cell containing the value of interest. Note the first cell in R_1 is #1, in R_2 it is #9 and in R_3 it is #25 etc, as shown in Fig. A.1.

Analysis for a$_2$.

a_2 is only affected by k and B. The values of a_2, determined from a series of term sequences in the spiral, for a number of values of k and B, are shown in the following table.

k	B	a_2
1	1	4
1	2	16
1	3	36
2	1	8
2	2	32
2	3	72
3	1	12
3	2	48
3	3	108

Table A.1 - Values of a_2 for k and B.

From Table A.1 it is clear that

$$a_2 = 4kB^2 \tag{A.1}$$

Analysis for a$_1$.

a_1 is affected by k, B, D, and R_n. The values of a_1 for various values of these parameters, determined from the spiral values of quadratic sequences, are shown in Table A.2 below. R_n here is the ring number containing the spiral cell in which the first quadratic term resides.

k	B	R_n	D	a_1	k	B	R_n	D	a_1	k	B	R_n	D	a_1
1	1	0	A	-3	1	1	1	A	4	2	1	1	A	10
			B	-2				B	6				B	12
			C	-1				C	7				C	14
			D	0				D	8				D	16
			E	1				E	9				E	18
			F	2				F	10				F	20
			G	3				G	12				G	22
			H	4				H	14				H	24
1	2	0	A	-6	1	2	1	A	10	2	2	2	A	52
			B	-4				B	12				B	56
			C	-2				C	14				C	60
			D	0				D	16				D	64
			E	2				E	18				E	68
			F	4				F	20				F	72
			G	6				G	22				G	76
			H	8				H	24				H	80
2	3	2	F	108	3	2	2	D	96	3	2	3	A	126
			G	114				E	102				B	132
			H	120				F	108				C	138

Table A.2 - Values of a_2 for k, $\underline{B}$, $\underline{R}_n$ and $\underline{D}$.

Note that some of these values will vary if the term sequence crosses the Base Number boundary. Analysis of Table A.2 shows that a_1 can be represented by

$$a_1 = (8R_n + D)\ kB \qquad (A.2)$$

Analysis for a$_0$.

a_0 is dependent upon k, P, R_n and n_r. Table A.3 below shows the values of a_0 for various values of these parameters as determined from the spiral.

k	R_n	n_r	P	a_0
1	1	5		5
1	1	7		7
1	1	9		9
2	1	3		5
2	2	9	1	33
2	2	14		43
3	2	4		34
3	2	10		52
3	2	15		67

Table A.3 - Values of a_0 for k, $\underline{R}_n$, $\underline{n}_r$ and $\underline{P}$.

Analysis of Table A.3 shows that

$$a_0 = \left(8\sum_{n=1}^{(n-1)}(R_n) + n_r - 1\right)k + P \qquad (A.3)$$

which via the standard formula for arithmetic series, becomes

$$a_0 = \left(4\left(R_n - 1\right)R_n + n_r - 1\right)k + P \tag{A.4}$$

Note that (A.4) can be used to determine the value of the contents of any cell in the spiral.

The final equation governing the generation of quadratics from Ulam's Spiral is, by combining (A.1), (A.2) and (A.4)

$$N = 4kB^2x^2 + \left(8R_n + D\right)kBx + \left(4\left(R_n - 1\right)R_n + n_r - 1\right)k + P \tag{A.5}$$

Thus if $k = 0.25$, $B = 1$, $R_n = 0$, (so that $n_r = 1$), $D = 4$, (Direction H) and $P = 41$, (A.5) becomes

$$N = x^2 + x + 41 \tag{A.6}$$

i.e. Euler's equation.

Similarly if $k = 0.5$, $B = 1$, $R_n = 0$, (so that $n_r = 1$), $D = 0$, (Direction D) and $P = 29$, (A.5) becomes

$$N = 2x^2 + 29 \tag{A.7}$$

Legendre's equation.

APPENDIX B.

Examples of Lining Up Sequence Terms in the Spiral.

There are two methods to achieve this, the iterative cell count method, and a semi-analytical method based upon the results of Appendix A. The second method is superior in that it reveals all conditions under which the terms of any particular quadratic will line up in the spiral. Either process can also show how this exercise can reveal additional prime generating quadratics.

B.1 The Iterative Cell Count Method

After maximisation of a quadratic, take the first four primes as generated from "Numbers Generated" on PrimeGen.xls, and enter them into the "Special Search Numbers" on Ulam.xls and "Re-Configure Primes". There are two possible outcomes.

  (i) The four primes in question will curl to the right in the spiral. In most cases lining up the complete sequence is merely a matter of inserting the smallest of the four primes in the "Base Number Input" cell and reconfiguring primes. However, where this still results in a curl to the right, it is necessary to count the smallest numbers of cells to individually move each prime number to a cell adjacent to the next largest. Cells must be counted clockwise and from a large number to a smaller. Add these counts, multiply by the "Step Size" and add the result to the existing "Base Number" to produce a new Base Number. Insert this into "Base Number Input" and reconfigure primes. The sequence should then line up somewhere in the spiral. In some cases it may be necessary to repeat this process a number of times.

  (ii) The four numbers in question will curl to the left. In this case the same process as in (i) should be followed but counting anti-clockwise from smaller numbers to larger.

This procedure is illustrated in the following examples.

B.1.1 - Reduced Equation Curling Clockwise.

This example shows how to line up reduced equations curling clockwise in the spiral, and also how this process can reveal additional prime generating quadratics.

  (i) In Ulam.xls, set the Base Number to 1, and the Step Size to 1.5. Press the "Reconfigure Primes" button.

  (ii) Select the sequence of primes, 109, 163, 229, 307 etc. and insert these numbers into the cells "Terms Numbers Input from Spiral".

  (iii) This yields the quadratic

$$N = 6x^2 + 48x + 109 \qquad \text{(B.1)}$$

in the "Equation Coefficient" cells.

  (iv) Insert the coefficients of (B.1) into the "Equation to be Reduced" cells of PrimeGen.xls, and vary the prime number in the cell marked "P" to find the lowest value which gives integer values in the "N+" and "N-" cells. The lowest value is 13 to give in the N+ cells

$$N = 6x^2 + 13 \qquad \text{(B.2)}$$

as the reduced equation.

  (v) Insert the coefficients of (B.2) into the "Coefficient Input" cells, ($a_1 = 0$). The sum of the coefficients is a prime number and the original four primes in (ii) are generated, preceded by the additional primes 13, 19, 37 and 67, and there are no composites present in this sequence. The reduced equation is therefore acceptable.

  (vi) Insert the first four primes in (v) in the "Special Number Search" cells in Ulam.xls and reconfigure primes. These four primes will be highlighted curling to the right in the spiral.

  (vii) Insert the lowest prime, 13, into the "Base Number Input" cell and reconfigure primes to yield 13 consecutive primes lying in direction D, showing that (B.2) is a Type (a) Full Prime quadratic.

Now note that the reconfigured spiral also yields another sequence of primes starting with 211 lying on the RHS of the spiral parallel to direction H. Taking the first four and proceeding as in (ii) and (iii) above yields

$$N = 6x^2 + 66x + 211 \qquad \text{(B.3)}$$

reducing this quadratic as in (iv) above yields

$$N = 6x^2 + 6x + 31 \qquad \text{(B.4)}$$

and which via (v) above produces the original four primes preceded by 31, 43, 67, 103 and 151. Processing the first four of these as in (vi) above again results in their curling clockwise in the reconfigured spiral. Thus inserting 31 as the Base Number, reconfiguring primes yields a long sequence of consecutive primes lying along direction H and extending beyond the periphery of the

spiral. Checking the generated terms in PrimeGen.xls proves (B.4) to be a Type (b) Full Prime quadratic generating 29 primes starting with 31 and ending with 4903.

B.1.2 - Reduced Equation Curling Anti-Clockwise.
Example 1 - Integer and Negative Values.

In Ulam.xls set the Base Number to -29 and the Step Size to 3. Select the sequence of primes 271, 409, 571 and 757 to produce

$$N = 12x^2 + 126x + 271 \tag{B.5}$$

Reducing (B.5) gives

$$12x^2 + 54x + 1 \tag{B.6}$$

which generates the above four primes preceded by 1, 67 and 157. When, with 271, these are reconfigured in the spiral, they curl anticlockwise. To line up the complete sequence, the minimum number of cell moves to bring each prime adjacent to the next largest is 2 each. Totalled this is 6, which when multiplied by the Step Size is 18. Subtract this from the existing Base Number, (-29) to give -47 and insert this as the new Base Number. The whole sequence of 12 primes now lines up parallel to direction F, Full Prime Type (c).

Example 2 - Fractional Values.

In Ulam.xls set the Base Number to 1 and the Step Size to 3.75. Select the sequence of primes 271, 421, 601 and 811 etc. to produce

$$N = 15x^2 + 135x + 271 \tag{B.7}$$

Reduced this becomes

$$N = 15x^2 + 45x + 1 \tag{B.8}$$

which produces the above four primes preceded by 1, 61 and 151. Together with 271 when reconfigured in the spiral they curl anti-clockwise. The total count to bring all four numbers adjacent is 6, multiplied by the Step Size is 22.5. Subtracted from the existing Base Number gives -21.5 as the new Base Number, resulting in 12 consecutive primes lying parallel to direction H, Full Prime, Type (c).

B.2 The Semi-Analytic Method.

This method is best described by example. Thus consider the second example in B.1.1 above, i.e. (B.4).

First substitute (A.1) into (A.2) for k to get

$$(8R_n + D) = 4\frac{a_1}{a_2}B \tag{B.9}$$

Inserting the coefficients of (B.4) into this gives

$$(8R_n + D) = 4B \tag{B.10}$$

Because the coefficient of B here is an integer, B will be unity. R_n is the integer result of dividing the RHS of (B.10) by 8, and D is the remainder. Thus $B = 1$, $R_n = 0$, and $D = 4$.
Now, knowing B, k can be determined from (A.1), thus

$$k = \frac{a_2}{4B^2} = 1.5 \tag{B.11}$$

Now, in order to determine P from (A.3), it is necessary to determine n_r, which in turn is determined from the geometry of the spiral. For integer values of D, n_r is given by

$$n_r = (4 + D) R_n + 1 \, {}^{+u}_{-l} \tag{B.12}$$

Here u and l are functions of D and are determined from Table B.1 at the end of this Appendix. Thus for this example n_r becomes

$$n_r = (4 + 4) \, 0 + 1 \, {}^{+0}_{-0} = 1 \, {}^{+0}_{-0} \tag{B.13}$$

So that from (A.3), P can now be determined as

$$p = a_0 - (4R_n (R_n - 1) R_n + n_r - 1) = 31 - \left(1 \, {}^{+0}_{-0} - 1\right) = 31 \tag{B.14}$$

Thus in the spiral, insertion of 31 for the Base Number and 1.5 as the Step Size, aligns all the terms along direction H.

When the coefficient of B in the RHS of (B.9) is fractional. B can take several values to align the applicable sequence terms in different directions in the spiral. This is demonstrated in the next example.

Consider the quadratic

$$N = 64x^2 + 18x + 1 \tag{B.15}$$

Here,

$$(8R_n + D) = 1.125B \tag{B.16}$$

(i) $B = 4$, so that $R_n = 1$ and $D = $ -3.5 and from (A.1), $k = 1$. Then from Table B.2 at the end of the Appendix

$$n_r = 2 \, {}^{+1}_{-0} \tag{B.17}$$

and therefore

$$P = 1 - \left(2 \, {}^{+2}_{-0} - 1\right) = -1 \ or \ 0 \tag{B.18}$$

and the terms line up in direction A in the spiral.

(ii) $B = 8$, so that $R_n = 1$, $D = 1$ and $k = 0.25$. Then n_r becomes $6 \, {}^{+1}_{-1}$, and P becomes -0.5 or 0. The terms then line up in direction E for both values of P, (only the first two terms are however visible because of the large value of B).

D	u	l
-3	R_n	R_n - 1
-2	$2R_n$	$2R_n$ - 1
-1	R_n	R_n
0	$2R_n$	$2R_n$
1	R_n	R_n
2	$2R_n + 1$	$2R_n$
3	$R_n + 1$	R_n
4	$2R_n + 2$	$2R_n$

D	n_r	u	l
-3.5	$R_n + 1$	R_n	R_n - 1
-2.5	$R_n + 1$	R_n	R_n - 1
-1.5	$3R_n + 1$	R_n	R_n
-0.5	$3R_n + 1$	R_n	R_n
0.5	$5R_n + 1$	R_n	R_n
1.5	$5R_n + 1$	R_n	R_n
2.5	$7R_n + 1$	R_n	$R_n + 1$
3.5	$7R_n + 1$	R_n	$R_n + 1$

<div align="center">

**Table B.1 - u and l as a
Function of R_n for Integer
Values of D**

**Table B.2 - n_r, u and l
as a Function of R_n
for Fractional Values of D.**

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Note that these values of u and l only apply for non-zero values of R_n, when R_n is zero, both u and l are also zero.

Finally, there are three directions in which the first two terms of a sequence can straddle the Base Number, (to identify this condition, the first two terms appear in the same spiral ring). Under this special condition the above two tables do not apply and n_r is given by

Direction F, $(D = 2)$:- $n_r = (4 + D)R_n + 1 \pm 0$

Direction E + ½, $(D = 1.5)$:-

$$n_r = 2R_n + 1 \pm 0$$

Direction F + ½, $(D = 2.5)$:-

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APPENDIX C.

Reduction Maximisation of Some Well known Sub-Prime Quadratics.

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In [2], there are listed some example polynomials that generate sequences of consecutive prime numbers. Of the 15 listed, four have already been discussed in this paper or the associated spreadsheet. A further three are of orders higher than two. That leaves eight which are discussed below.

(i) 36n^2 - 810n + 2733, (Type (c)).

This quadratic is quoted in [2] as generating 45 distinct primes for $n = 0$ to 44. However, the sequence is

  (i) 2753, 1979, 1277, 647, 89 = 5 primes

 (ii) -397 to -163 = 18 negative numbers, (13 unique)

(iii) 359 to 36809 = 27 primes.

Consequently the claim for 45 distinct primes must include the 13 unique negative numbers. Although negative numbers can conform to the definition of primeness, their inclusion in the group is considered a dubious step and needs clarification by an appropriate authority.

To generate only the positive primes above, two quadratic's are needed. Using the reduction process of this paper, they are

$$N = 36n^2 + 558n + 359 \tag{C.1}$$

Generates 359 to 36809, (27 primes)

and

$$N = 36n^2 + 522n + 89 \tag{C.2}$$

Generates 89 to 2753, (5 Primes).

(ii) $47n^2$ - 1701n + 10181, (Type (c)).

This quadratic is quoted in [2] as generating 43 distinct primes for $n = 0$ to 42. However, it exhibits the same characteristics as (C.1) above. The sequences are

(i) 10181 to 577 = 8 primes

(ii) -419 to -599 = 21 negative numbers, (all unique)

(iii) 379 to 21647 = 14 primes

To generate only the positive primes above, two quadratics are needed as follows

$$N = 47n^2 + 1025n + 379 \tag{C.3}$$

Generates 379 to 21647, (14 primes)

and

$$N = 47n^2 + 1043n + 577 \tag{C.4}$$

Generates 577 to 10181, (8 primes)

(iii) $103n^2$ - 4707n + 50383, (Type (c)).

In [2] this quadratic is quoted as generating 43 primes for $n=$ 0 to 42. Again it exhibits the same characteristics as (C.1) and (C.2). The sequences generated are

(i) 50383 to 131 = 18 primes

(ii) -971 to -1619 = 11 negative numbers, (all unique)

(iii) 503 to 34381 = 14 primes.

To generate only the positive primes above, two quadratics are needed as follows

$$N = 103n^2 + 1205n + 131 \tag{C.5}$$

Generates 131 to 50383, (18 primes).

and

$$N = 103n^2 + 1267n + 503 \qquad\qquad (C.6)$$

Generates 503 to 34381, (14 primes).

(iv) $43n^2$ - 537n + 2971, (Type (c)).

This equation is quoted as generating 35 primes for $n = 0$ to 34. It is an anti-symmetric V-Prime equation generating the following sequences.

(i) 2971 to 1297 = 7 primes

(ii) 1319 to 34421 = 28 primes.

To generate each separate arm in ascending order, two quadratics are needed as follows.

$$N = 43n^2 + 21n + 1297 \qquad\qquad (C.7)$$

Generates 1297 to 2971, (7 primes).

and

$$N = 43n^2 + 65n + 1319 \qquad\qquad (C.8)$$

Generates 1319 to 34421, (28 primes).

(v) $8n^2$ - 488n + 7243, (Type (c)).

This equation is quoted as generating 31 distinct primes for $n = 0$ to 61. It is a symmetrical V-Prime equation. It generates the following sequences.

(i) 7243 to 43 = 26 primes.

(ii) -37 to -197, -197 to -37 = 10 negative numbers, (5 unique).

(iii) 43 to 7243 = 26 primes.

The 31 primes therefore comprises (i) or (iii) plus the five unique negative numbers. To generate only the sequence of positive primes in ascending order, the following quadratic suffices.

$$8n^2 + 88n + 43 \qquad\qquad (C.9)$$

Generates 43 to 7243, (26 primes).

(vi) $4n^2$ + 4n + 59, (Type (b)).

This is a Sub-Prime quadratic quoted in [2] as generating 14 primes. It is already maximised.

(vii) $7n^2$ - 371n + 4871, (Type (c))

This quadratic is quoted in [2] as generating 24 distinct primes for $n = 0$ to 23. It is a symmetrical V-Prime equation. It generates the following sequences.

(i) 4871 to 41 = 24 primes.

  (ii) -1 to -43, -43 to -1 = 6 negative numbers, (3 unique).

  (iii) 41 to 4871 = 24 primes.

To generate only the positive primes in ascending order, one equation is required thus

$$N = 7n^2 + 49n + 41 \tag{C.10}$$

Generates 41 to 4871, (24 primes).

(viii) $6n^2$ - 342n + 4903, (Type (c)).

This equation is quoted in [2] as generating 29 primes for $n = 0$ to 28. It is a symmetrical V-Prime equation. To generate just the ascending sequence of primes, the following equation applies

$$6n^2 + 6n + 31 \tag{C.11}$$

Generates 31 to 4903, (29 primes).

and is therefore a Full Prime equivalent, (see the second half of Appendix B.1).

All of the reduced equations produced here can be displayed in aligned form in the spiral. However, to do this using the first method, there are two possible difficulties to overcome due to the fact that the original quadratics were not obtained from the spiral with the correct Step Size or Base Number. The Step Size can, in some cases, only be obtained by trial and error from (A.1). Secondly, having obtained the correct Step Size, alignment of terms will need several iterations of the process described and demonstrated in Appendix B. The second method is consequently the better one to obtain a quick result.

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[1] P.G.Bass, *Prime Numbers, Generation by Single Value Polynomial Equations,* February 2009, www.relativitydomains.com.

[2] Eric Weinstein, (Wolfram Research), *Prime Generating Polynomial,* 8 February 2009, www.mathworld.com.