PRIME NUMBERS

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GENERATION BY SINGLE VARIABLE POLYNOMIAL EQUATIONS

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ABSTRACT

The purpose of this paper is to identify, under a strict set of control conditions, those single variable polynomial equations of order 2 or higher, that generate only prime numbers.

1 Introduction.

Since their discovery, prime numbers have been the subject of intensive study by both professional and amateur mathematicians alike. The nature of these studies has ranged from sequence characteristics through general distribution within the complete set of integer numbers, through to their generation by a variety of equation types.
With regard to the latter, Legendre proved that there is no rational algebraic function in a variable n which would generate all the primes for all n. Similarly, Goldbach proved in 1752 that no polynomial with integer coefficients can generate even a subset of primes for all n. Consequently, study in this area has concentrated on finding those equations which give a sub-set of all primes for some n.
The primary purpose of this paper is to determine, under a specific set of control criteria, those single variable polynomial equations of order 2 or higher, that will generate only prime numbers for a range of integer values of the single variable n from 0 upwards.
The initial secondary objective is to find those equations which as a group combine to generate all the 26 prime numbers from 0 to 100. This is followed by an invitation to readers to participate in finding additional equations to generate all the prime numbers from 101 to 200.

2 Prime Number Generation.

2.1 Preamble and Establishment of Control Conditions.

The first matter is consideration of the number zero. Zero appears in all branches of science and as such has long been accepted as a member of the complete set of numbers. Insofar as it lies equidistant between -1 and +1 determines that it must also be an integer. The remaining question is whether it is prime.
According to the formal definition of a prime number, viz. any number that can only be divided by itself and unity, zero would not classify as prime for the following reasons.
Albeit that

$$\frac{0}{0} \tag{2.1}$$

is indeterminate, taking

$$\frac{0}{1} = 0 \tag{2.2}$$

any other number divided into zero gives

$$\frac{0}{n} = 0 \tag{2.3}$$

the same value as (2.2) and therefore, apart from (2.1), zero can be divided by any and all numbers, rational and irrational, to give an integer value. Zero is the only number that possesses this characteristic. Accordingly, zero is not prime but, must be included as an integer value, the first, in the single variable n in the polynomials to be determined.
The second matter to be discussed is the control criteria under which these polynomials are to be obtained. Representing these equations thus,

$$N = a_m n^m + a_{(m-1)} n^{(m-1)} + \cdots + a_0 \tag{2.4}$$

The criteria to be set against suitable variants of (2.3) are threefold as follows

(i) The coefficients a_i must all be real and integer.

(ii) Clearly, when in (2.4) $n = a_0, N$ must be composite. The criterion to be applied to n is therefore that all equations must generate only prime numbers for all values of n from zero up to at least (a_0-2), (this ensures that all equations will generate at least (a_0-1) prime numbers). In addition, all equations must generate at least four prime numbers.

(iii) When $n = 0$, $N = a_0$ so that a_0 itself must be a prime number.

2.2 A Brief Resume of Existing Equations.

A great many equations have been found that generate a small range of prime numbers, but only a few conform to the criteria above. The most famous of these is that found by Euler in 1772, viz.

$$N = n^2 + n + 41 \tag{2.5}$$

Equation (2.5) generates only prime numbers for $n = 0$ to 39. There are two variants of (2.5) that conform to the criteria, the first is

$$N = n^2 + n + 11 \tag{2.6}$$

which generates prime numbers for $n = 0$ to 9. The second variant is

$$N = n^2 + n + 17 \tag{2.7}$$

This equation generates prime numbers for $n = 0$ to 15.
A further equation discovered by Legendre is

$$N = 2n^2 + 29 \tag{2.8}$$

which generates prime numbers for $n = 0$ to 28, and a variant is

$$N = 2n^2 + 11 \tag{2.9}$$

This generates prime numbers for $n = 0$ to 10.

The prime numbers generated by the above five equations, (2.5) to (2.9), are shown in the table below. Note that the order of listing is 1^{st} by a_0, 2^{nd} by a_1, and 3^{rd} by a_2.

Equation	Prime Numbers Generated	
	Unique	Repeats
$N = 2n^2 + 11$	11, 13, 19, 29, 43, 61, 83, 109, 139, 173, 211	
$N = n^2 + n + 11$	17, 23, 31, 41, 53, 67, 101	11, 13, 83
$N = n^2 + n + 17$	37, 47, 59, 73, 89, 107, 127, 149, 199, 227, 257	17, 19, 23, 29, 173
$N = 2n^2 + 29$	79, 157, 191, 229, 271, 317, 367, 421, 479, 541, 607, 667, 751, 829, 911, 997, 1087, 1181, 1279, 1381, 1487, 1597	29, 31, 47, 61, 101, 127
$N = n^2 + n + 41$	41, 71, 97, 113, 131, 151, 197, 223, 251, 281, 313, 347, 383, 461, 503, 547, 593, 641, 691, 743, 797, 853, 971, 1033, 1097, 1163, 1231, 1301, 1373, 1447, 1523, 1601	43, 47, 53, 61, 83, 173, 421, 911

Table 2.1 - Prime Numbers Generated By Equations (2.5) to (2.9).

It should be noted here that as more equations are identified and inserted into the Table according to the listing order above, the distribution between "Unique" and "Repeats" in this Table will change. Some equations may thereby become redundant and removed to be added to the redundant list in Appendix A.

2.3 Equations to Complete the Set Required to Generate the First 26 Prime Numbers Between 0 and 100.

Reference to Table 2.1 shows that there are only five numbers not covered by the equations listed. They are 1, 2, 3, 5, and 7, i.e. the first five. Initially only quadratic equations are being considered so that it is only 1, 3, 5, and 7 that need to be considered here. This is because the prime number 2 cannot be generated by a quadratic, except when $n = 0$, but which thereafter does not conform to the criteria set.

There are two equations required to generate the above four primes, the first is

$$N = n^2 + n + 1 \tag{2.10}$$

which is clearly a simple variant of Euler's equation, (2.5). The second equation is

$$N = n^2 + 3n + 1 \tag{2.11}$$

Eq, (2.10) generates 1, 3 and 7, (plus repeat 13), while (2.11) generates 5, (plus repeats 1, 11, 19, 29 and 41). Thus together with (2.5) to (2.9), (2.10) and (2.11) complete the generation of all the 26 prime numbers between 0 and 100.

2.4 Equation Categorization and Inter-Conversion.

With the quadratics of the previous two Sections and other similar equations in the literature, (plus additionally those developed via preparation of this article), it is now possible to categorize these equations and show how they can be inter-converted. This will help in the search for more.

2.4.1 Categorization.

Type (i) - Full Prime Generators. - These are equations that generate prime numbers exactly according to the criteria set in Section 2.1. An example is Euler's equation, (2.5).

Type (ii) - Sub Prime Generators. - These are equations that generate prime numbers for some initial part of the criteria concerning the variable n but then fail. An example is

$$N = 2n^2 + 4n + 31 \tag{2.12}$$

This generates prime numbers for $n = 0$ to 3 and then fails.

Type (iii) - Symmetrical V Prime Generators. - These are equations that generate prime numbers in descending order for $n = 0$ to ($a_0/2$ - 1), and then the same prime numbers in ascending order for $n = a_0/2$ to (a_0 - 2). They do not meet the criteria of Section 2.1. An example is

$$N = n^2 - 79n + 1601 \tag{2.13}$$

Type (iv) - Non-Symmetrical V Prime Generators. - These are equations that generate prime numbers in descending order for $n = 0$ to n_i, and then a greater or lesser number of prime numbers in ascending order from $n = n_{i+1}$ to n_{i+m}. They do not conform to the criteria.

2.4.2 Inter-Conversion of Equation Categories.

From Type (ii), Sub Prime to Type (i), Full Prime. - In some cases a Sub Prime equation will generate a sub-set of a Full Prime equation. The following process will test any potential Sub Prime equation for this and make the conversion. Consider a generalised Sub Prime equation thus

$$N = a_2n^2 + a_1n + a_0 \tag{2.14}$$

If the converted Full Prime equation is obtained by putting $n = n$ - b, where b is some integer to be determined, then

$$N = a_2 (n - b)^2 + a_1 (n - b) + a_0 \tag{2.15}$$

which expands to

$$N = a_2n^2 - (2a_2b - a_1)\ n + a_2\, b^2 - a_1b + a_0 \tag{2.16}$$

If (2.16) is a Full Prime Generator, then, in accordance with the criteria, the constant term must be a prime number P, so that

$$a_2b^2 - a_1b + a_0 - P = 0 \tag{2.17}$$

Solving for b

$$b = \frac{a_1 \pm \left\{ a_1^2 - 4a_2 (a_0 - P)\right\}^{1/2}}{2a_2} \tag{2.18}$$

The test/conversion process is now completed by finding the lowest prime value for P that makes b an integer. This process is simply one of moving the N axis along the n axis to the lowest prime on the curve to maximise the number of primes generated. However, it should be noted that even though a value for P can be found, the resulting polynomial may not be Full Prime.

Example.
Consider the Sub-Prime equation

$$N = 2n^2 + 48n + 317 \tag{2.19}$$

which generates primes starting with 317 for $n = 0$, but does not meet the criteria. With $a_2 = 2$, $a_1 = 48$ and $a_0 = 317$ substituted in (2.18) there results

$$b = \frac{48 \pm \sqrt{-232 + 8P}}{4} \tag{2.20}$$

The smallest prime value of P to make b an integer is 29, with which (2.20) becomes

$$b = 12 \tag{2.21}$$

Substituting this into (2.16) finally gives

$$N = 2n^2 + 29 \tag{2.22}$$

which is Legendre's equation and is Full Prime.

From Types (iii) and (iv), V prime to Full prime. - V Prime equations can also be converted to Full Prime via an identical process to that above with the minor difference that the catalyst term is $n + b$. In that case the equivalent of (2.16) is

$$N = a_2 n^2 + (2a_2 b - a_1)\ n + a_2\, b^2 - a_1 b + a_0 \tag{2.23}$$

Note that the constant term in (2.23) is identical to that in (2.16) so that the test/conversion equation of (2.18) applies in this case also.

Example.
Consider the Symmetrical V Prime Generator

$$N = n^2 - 79n + 1601 \tag{2.24}$$

With $a_2 = 1$, $a_1 = 79$ and $a_0 = 1601$ substituted in (2.18) there results

$$b = \frac{79 \pm \sqrt{-163 + 4P}}{2} \tag{2.25}$$

With $P = 41$ this becomes

$$b = 40 \quad or \quad 39 \tag{2.26}$$

Substituting $b = 40$ into (2.23) gives

$$N = n^2 + n + 41 \tag{2.27}$$

which is Euler's equation and is Full Prime. Note that substituting $b = 39$ into (2.23) gives

$$N = n^2 - n + 41 \tag{2.28}$$

which is Legendre's version of Euler's equation and is Non-Symmetrical V Prime. Applying the conversion process to (2.28) gives

$$b = 1 \quad or \quad 0 \tag{2.29}$$

With $b = 1$, (2.28) is converted to (2.27) and with $b = 0$, (2.28) sensibly does not change.
The categorisation and conversion process should be of some assistance in the challenge of the next section.

2.5 The Challenge and the PRIMES Program.

It can be seen from Table 2.1 and the previous Section, that not only have the first 26 prime numbers between 0 and 100 been generated by Eqs. (2.5) to (2.11), but also the majority of those between 101 and 200. Those not covered in this latter range are:- 101, 103, 137, 163, 167, 179, 181 and 193.

Finding the equations, which, while conforming to the criteria set in Section 2.1, will generate the above numbers, (as well as repeats and others), is left as a challenge for any interested reader. Equations so found should be forwarded via the email address on the LINKS page of this website, and will be included in the next update of this article, together with an acknowledgement.
To assist in this task, a Visual Basic program, PRIMES, can be downloaded and used to check the numbers generated by any potential equation for primeness. The program will check any single number for primeness, (up to a maximum of 2,147,483,647), and/or generate all the prime numbers within any range of 9,000 up to the same limit. The PRIMES program can be downloaded from the website introduction page for this article, (www.relativitydomains.com/mathematics/primes).

In participating in this exercise, apart from the criteria, there are a some significant points to note.

(i) There are many equations which generate prime numbers which duplicate those already covered above. Some of these may also generate new numbers albeit not those that are the subject of this challenge. Such equations should also be forwarded for inclusion in the next or subsequent update.

(ii) Only quadratic equations should be considered. Higher orders will be the subject of future updates.

(iii) While a manual process is quite feasible in conducting this exercise, it would be both very laborious and tedious, and therefore for those who wish to participate it would be preferable to adopt some form of computer analysis.

(iv) It is already known that there are no acceptable equations with a greater a_0 than in (2.5) and (2.8) for those particular types of equations. It is not known whether this limit applies to all quadratic equations constructed for this purpose. However, note that the higher the value of a_0, the more primes will need to be generated in order for the criteria to be met.

(v) There are at most seven equations needed to generate the above eight primes.

3 A Simple Method of Equation Generation.

This spreadsheet based computerised methodology will be included in the next update to this article.

4 Conclusions.

The criteria that have been adopted here severely restricts the number of acceptable equations. However, it does ensure that all equations that conform, present a highly standardised set of characteristics. Also, working to cover all prime numbers from zero upwards, provides a logical and systematic approach that may subsequently result in equation patterns that will aid in further

identification.
The goal that has been set in this initial paper is a very limited one, but it is only a start. It is not known whether polynomial equations of the type utilised here can eventually generate prime numbers ad infinitum, but as progress materialises it is hoped that future targets will become more ambitious and challenging.

APPENDIX A.

Redundant Equations.

A.1 Repeaters.

These are quadratics that, while meeting all the criteria as delineated in the main text, only generate primes already obtained from the equations in Sections 2.2 and 2.3. A list of those known to date will be added in the next update.

A.2 Sub Prime Equations.

Sub Prime Equation	Converts To
$2n^2 + 4n + 31$	$2n^2 + 29$
$2n^2 + 48n + 317$	$2n^2 + 29$

This list will be expanded in the next update.

A.3 V Prime Equations.

V Prime Equation	Converts To
$n^2 - 19n + 101$	$n^2 + n + 11$
$n^2 - 31n + 257$	$n^2 + n + 17$
$n^2 - 79n + 1601$	$n^2 + n + 41$

This list will be expanded in the next update.

REFERENCES.

[1] David Wells, *Curious and Interesting Numbers,* Penguin Books, 1988

[2] Online, *Formula for Primes,* www.wikipedia.com.

[3] Online, *Prime Formulas,* www.mathworld.com.

[4] Online, *Prime Generating Polynomial,* www.mathworld.com.