PRECISE DETERMINATION OF THE MULTIPLE ROOTS OF HIGH ORDER POLYNOMIALS.

(1)

THE DIFFERENTIAL DIVISION REMAINDER METHOD.

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ABSTRACT

This paper presents a new method for the precise determination of the multiple roots of polynomial equations of any order, n, greater than 2. The method, the Differential Division Remainder Method, is applicable when any polynomial contains two roots in a variety of multiple root configurations.

1 INTRODUCTION.

There are many methods of determining the roots of polynomial equations, the majority of which involve some form of iterative process. They all have varying degrees of success when the polynomial contains only singular roots, or at most a pair of multiple roots, the rest being singular. However, when the polynomial contains a high number of multiple roots, in a variety of configurations, iterative methods all experience some difficulty in obtaining a solution with any degree of precision. This is because such roots do not cross the abscissa, but lie tangential to it, and a process of finite iteration cannot determine the precise point of contact.

The method of determination presented here, avoids this difficulty because it does not utilise an iterative approach, and is purely analytic in nature. It is termed the Differential Division Remainder Method, and is based upon the fact that if a polynomial is differentiated with respect to its independent variable, and the differential is then divided into the original equation, the remainder term ratios are quadratic functions of the primary multiple root. The process is an extension of one briefly described in [2].

This extended method is applicable to polynomials of any order where they contain two roots in a variety of multiple configurations. Also, irrespective of the order of the polynomial, implementation of the method is easily, and relatively quickly, effected via manual computation. Both manual and computer implementation is discussed here.

2 Description of the Method.

2.1 Preamble.

The polynomial root configurations to which this method is applicable, for equations of order 3 to 10, is illustrated in Table 2.1 below. In this paper the most prolific root is termed the primary root, r_p, with all others the secondary root(s), r_s.

Order	Configuration of Roots ($r_p + r_s$)
3	3 + 0 2 + 1
4	4 + 0 3 + 1 2 + 2 2 + 1 + 1
5	5 + 0 4 + 1 3 + 2 3 + 1 + 1
6	6 + 0 5 + 1 4 + 2 4 + 1 + 1 3 + 3
7	7 + 0 6 + 1 5 + 2 5 + 1 + 1 4 + 3
8	8 + 0 7 + 1 6 + 2 6 + 1 + 1 5 + 3 4 + 4
9	9 + 0 8 + 1 7 + 2 7 + 1 + 1 6 + 3 5 + 4
10	10 + 0 9 + 1 8 + 2 8 + 1 + 1 7 + 3 6 + 4 5 + 5

Table 2.1 - Root Configurations Applicable to the

Differential Division Remainder Method.

The configuration $r_p + 0$ is included here as it completes the set and is conducive to the same general analysis.

Thus for polynomials of orders 3 to 10, a total of 39 root configurations can be analysed using this method. These are the only configurations for which the remainder term ratios from the DDR process are quadratic functions of the primary root. For other configurations, three or more sets of multiple plus single roots, the remainder terms are cubic and higher functions of the primary root and therefore not easily solvable.

2.2 Development of Specific Algorithms for Polynomial Orders of 3 to 10.

The simplest way of illustrating this method is to consider a generalised 7^{th} order polynomial containing 4 primary roots, $x = -r_p$, and 3 secondary roots, $x = -r_s$. Writing this polynomial out in full gives

$$y = x^7 + (4r_p + 3r_s)\, x^6 + \left(6r_p^2 + 12r_pr_s + 3r_s^2\right) x^5 + \left(4r_p^3 + 18r_p^2r_s + 12r_pr_s^2 + r_s^3\right) x^4 + \left(r_p^4 + 12r_p^3r_s + 18r_p^2r_s^2 + 4r_pr_s^3\right) x^3 + \left(3r_p^4r_s + 12r_p^3r_s^2 + 6r_p^2r_s^3\right) x^2 + \left(3r_p^4r_s^2 + 4r_p^3r_s^2\right) x + r_p^4r_s^3 \tag{2.1}$$

Writing this as

$$y = x^7 + Ax^6 + Bx^5 + Cx^4 + Dx^3 + Ex^2 + Fx + G \tag{2.2}$$

its differential with respect to x is then

$$y^{/} = 7x^6 + 6Ax^5 + 5Bx^4 + 4Cx^3 + 3Dx^2 + 2Ex + F \tag{2.3}$$

Dividing (2.3) into (2.2) produces the following remainder terms

$$\begin{aligned}
R_5 &= \frac{2}{7}B - \frac{6}{49}A^2 \quad coefficient\ of\ the\ x^5\ remainder.\\
R_4 &= \frac{3}{7}C - \frac{5}{49}AB \quad coefficient\ of\ the\ x^4\ remainder.\\
R_3 &= \frac{4}{7}D - \frac{4}{49}AC \quad coefficient\ of\ the\ x^3\ remainder.\\
R_2 &= \frac{5}{7}E - \frac{3}{49}AD \quad coefficient\ of\ the\ x^2\ remainder.\\
R_1 &= \frac{6}{7}F - \frac{2}{49}AE \quad coefficient\ of\ the\ x\ remainder.\\
R_0 &= G - \frac{1}{49}AF \quad constant\ remainder.
\end{aligned} \tag{2.4}$$

Only the first three remainder terms, R_5, R_4 and R_3 are of interest initially. Substituting from (2.1) for A, B, C and D in these terms gives

$$\begin{aligned}
R_5 &= -\frac{12}{49}\left(r_p^2 - 2r_pr_s + r_s^2\right)\\
R_4 &= -\frac{12}{49}\left(3r_p^3 - 4r_p^2r_s - r_pr_s^2 + 2r_s^3\right)\\
R_3 &= -\frac{12}{49}\left(3r_p^4 - 8r_p^2r_s^2 + 4r_pr_s^3 + r_s^4\right)
\end{aligned} \tag{2.5}$$

So that the remainder term ratios are

$$\frac{R_4}{R_5} = 3r_p + 2r_s$$
$$\frac{R_3}{R_5} = 3r_p^2 + 6r_pr_s + r_s^2 \tag{2.6}$$

From the first part in (2.6)

$$r_s = \frac{1}{2}\left(\frac{R_4}{R_5} - 3r_p\right) \tag{2.7}$$

Substituting (2.7) into the second part of (2.6) produces the following quadratic

$$r_p^2 - \frac{2}{5}\frac{R_4}{R_5}r_p - \frac{1}{15}\left(\frac{R_4}{R_5}\right)^2 + \frac{4}{15}\frac{R_3}{R_5} = 0 \tag{2.8}$$

From which, using the standard formula for quadratics

$$r_p = \frac{1}{5}\frac{R_4}{R_5} \pm \left[\frac{8}{75}\left(\frac{R_4}{R_5}\right)^2 - \frac{4}{15}\frac{R_3}{R_5}\right]^{1/2} \tag{2.9}$$

Verification that this is indeed a root of the equation in question is then obtained by dividing (2.9) into the original equation, (2.1), to obtain a zero remainder. Subsequent to such verification, the secondary root is obtained, in this case, from (2.2) and (2.9) as

$$r_s = \frac{1}{3}\left(A - 4r_p\right) \tag{2.10}$$

Note that in (2.9) the positive square root gives the value of r_p when $r_p > r_s$ and vice versa for the negative square root.

This analysis applied to all the configurations of Table 2.1 produces the following Table of algorithms for their primary and secondary roots

Order	Root Configuration	Root Size Orientation	Primary Root Algorithm	Secondary Root Algorithm	Note
3	3 + 0	-	$A/3$	-	
	2 + 1	Both	R_0/R_1	$A{-}2r_p$	
4	4 + 0	-	$A/4$	-	
	3 + 1	Both	$\frac{1}{2}\frac{R_1}{R_2}$	$A{-}3r_p$	
	2 + 2	-	$\frac{1}{2}\frac{R_1}{R_2} \pm \left[\frac{1}{4}\left(\frac{R_1}{R_2}\right)^2 - \frac{R_0}{R_2}\right]^{1/2}$	$\frac{1}{2}(A - 2r_p)$	1
	2 + 1 +1	2>1>1 and 1>2>1	$\frac{1}{2}\frac{R_1}{R_2} + \left[\frac{1}{4}\left(\frac{R_1}{R_2}\right)^2 - \frac{R_0}{R_2}\right]^{1/2}$	See Note 3	2, 3
		1>1>2	$\frac{1}{2}\frac{R_1}{R_2} - \left[\frac{1}{4}\left(\frac{R_1}{R_2}\right)^2 - \frac{R_0}{R_2}\right]^{1/2}$		
5	5 + 0	-	$A/5$	-	
	4 + 1	Both	$\frac{1}{3}\frac{R_2}{R_3}$	$A{-}4r_p$	
	3 + 2	3 > 2	$\frac{1}{3}\frac{R_2}{R_3} + \left[\frac{1}{9}\left(\frac{R_2}{R_3}\right)^2 - \frac{1}{3}\frac{R_1}{R_3}\right]^{1/2}$	$\frac{1}{2}(A - 3r_p)$	
		2 > 3	$\frac{1}{3}\frac{R_2}{R_3} - \left[\frac{1}{9}\left(\frac{R_2}{R_3}\right)^2 - \frac{1}{3}\frac{R_1}{R_3}\right]^{1/2}$		
	3 + 1 + 1	3>1>1 and 1>3>1	$\frac{1}{3}\frac{R_2}{R_3} + \left[\frac{1}{9}\left(\frac{R_2}{R_3}\right)^2 - \frac{1}{3}\frac{R_1}{R_3}\right]^{1/2}$	See Note 4	2, 4
		1>1>3	$\frac{1}{3}\frac{R_2}{R_3} - \left[\frac{1}{9}\left(\frac{R_2}{R_3}\right)^2 - \frac{1}{3}\frac{R_1}{R_3}\right]^{1/2}$		
6	6 + 0	-	$A/6$	-	

Order	Root Configuration	Root Size Orientation	Primary Root Algorithm	Secondary Root Algorithm	Note
6	5 + 1	Both	$\frac{1}{4}\frac{R_3}{R_4}$	$A-5r_p$	
	4 + 2	4>2	$\frac{1}{4}\frac{R_3}{R_4}+\left[\frac{1}{16}\left(\frac{R_3}{R_4}\right)^2-\frac{1}{6}\frac{R_2}{R_4}\right]^{1/2}$	$\frac{1}{2}\left(A-4r_p\right)$	
		2>4	$\frac{1}{4}\frac{R_3}{R_4}-\left[\frac{1}{16}\left(\frac{R_3}{R_4}\right)^2-\frac{1}{6}\frac{R_2}{R_4}\right]^{1/2}$		
	4 + 1 + 1	4>1>1 and 1>4>1	$\frac{1}{4}\frac{R_3}{R_4}+\left[\frac{1}{16}\left(\frac{R_3}{R_4}\right)^2-\frac{1}{6}\frac{R_2}{R_4}\right]^{1/2}$	See Note 5	2, 5
		1>1>4	$\frac{1}{4}\frac{R_3}{R_4}-\left[\frac{1}{16}\left(\frac{R_3}{R_4}\right)^2-\frac{1}{6}\frac{R_2}{R_4}\right]^{1/2}$		
	3 + 3	-	$\frac{1}{4}\frac{R_3}{R_4}\pm\left[\frac{3}{16}\left(\frac{R_3}{R_4}\right)^2-\frac{1}{2}\frac{R_2}{R_4}\right]^{1/2}$	$\frac{1}{3}\left(A-3r_p\right)$	1
7	7 + 0	-	$A/7$	-	
	6 + 1	Both	$\frac{1}{5}\frac{R_4}{R_5}$	$A-6r_p$	
	5 + 2	5>2	$\frac{1}{5}\frac{R_4}{R_5}+\left[\frac{1}{25}\left(\frac{R_4}{R_5}\right)^2-\frac{1}{10}\frac{R_3}{R_5}\right]^{1/2}$	$\frac{1}{2}\left(A-5r_p\right)$	
		2>5	$\frac{1}{5}\frac{R_4}{R_5}-\left[\frac{1}{25}\left(\frac{R_4}{R_5}\right)^2-\frac{1}{10}\frac{R_3}{R_5}\right]^{1/2}$		
	5 + 1 +1	5>1>1 and 1>5>1	$\frac{1}{5}\frac{R_4}{R_5}+\left[\frac{1}{25}\left(\frac{R_4}{R_5}\right)^2-\frac{1}{10}\frac{R_3}{R_5}\right]^{1/2}$	See Note 6	2, 6
		1>1>5	$\frac{1}{5}\frac{R_4}{R_5}-\left[\frac{1}{25}\left(\frac{R_4}{R_5}\right)^2-\frac{1}{10}\frac{R_3}{R_5}\right]^{1/2}$		
	4 + 3	4>3	$\frac{1}{5}\frac{R_4}{R_5}+\left[\frac{8}{75}\left(\frac{R_4}{R_5}\right)^2-\frac{4}{15}\frac{R_3}{R_5}\right]^{1/2}$	$\frac{1}{3}\left(A-4r_p\right)$	

Order	Root Configuration	Root Size Orientation	Primary Root Algorithm	Secondary Root Algorithm	Note
7		3>4	$\frac{1}{5}\frac{R_4}{R_5}-\left[\frac{8}{75}\left(\frac{R_4}{R_5}\right)^2-\frac{4}{15}\frac{R_3}{R_5}\right]^{1/2}$	$\frac{1}{3}\left(A-4r_p\right)$	
8	$8+0$	-	$A/8$	-	
	$7+1$	Both	$\frac{1}{6}\frac{R_5}{R_5}$	$A{-}7r_p$	
	$6+2$	$6>2$	$\frac{1}{6}\frac{R_5}{R_6}+\left[\frac{1}{36}\left(\frac{R_5}{R_6}\right)^2-\frac{1}{15}\frac{R_4}{R_6}\right]^{1/2}$	$\frac{1}{2}\left(A-6r_p\right)$	
		$2>6$	$\frac{1}{6}\frac{R_5}{R_6}-\left[\frac{1}{36}\left(\frac{R_5}{R_6}\right)^2-\frac{1}{15}\frac{R_4}{R_6}\right]^{1/2}$		
	$6+1+1$	$6>1>1$ and $1>6>1$	$\frac{1}{6}\frac{R_5}{R_6}+\left[\frac{1}{36}\left(\frac{R_5}{R_6}\right)^2-\frac{1}{15}\frac{R_4}{R_6}\right]^{1/2}$	See Note 7	2, 7
		$1>1>6$	$\frac{1}{6}\frac{R_5}{R_6}-\left[\frac{1}{36}\left(\frac{R_5}{R_6}\right)^2-\frac{1}{15}\frac{R_4}{R_6}\right]^{1/2}$		
	$5+3$	$5>3$	$\frac{1}{6}\frac{R_5}{R_6}+\left[\frac{15}{216}\left(\frac{R_5}{R_6}\right)^2-\frac{1}{6}\frac{R_4}{R_6}\right]^{1/2}$	$\frac{1}{3}\left(A-5r_p\right)$	
		$3>5$	$\frac{1}{6}\frac{R_5}{R_6}-\left[\frac{15}{216}\left(\frac{R_5}{R_6}\right)^2-\frac{1}{6}\frac{R_4}{R_6}\right]^{1/2}$		
	$4+4$	-	$\frac{1}{6}\frac{R_5}{R_6}\pm\left[\frac{5}{36}\left(\frac{R_5}{R_6}\right)^2-\frac{1}{3}\frac{R_4}{R_6}\right]^{1/2}$	$\frac{1}{4}\left(A-4r_p\right)$	1
9	$9+0$	-	$A/9$	-	
	$8+1$	Both	$\frac{1}{7}\frac{R_6}{R_7}$	$A{-}8r_p$	
	$7+2$	$7>2$	$\frac{1}{7}\frac{R_6}{R_7}+\left[\frac{1}{49}\left(\frac{R_6}{R_7}\right)^2-\frac{1}{21}\frac{R_5}{R_7}\right]^{1/2}$	$\frac{1}{2}\left(A-7r_p\right)$	
		$2>7$	$\frac{1}{7}\frac{R_6}{R_7}-\left[\frac{1}{49}\left(\frac{R_6}{R_7}\right)^2-\frac{1}{21}\frac{R_5}{R_7}\right]^{1/2}$		

Order	Root Configuration	Root Size Orientation	Primary Root Algorithm	Secondary Root Algorithm	Note
9	$7+1+1$	$7>1>1$ and $1>7>1$	$\frac{1}{7}\frac{R_6}{R_7}+\left[\frac{1}{49}\left(\frac{R_6}{R_7}\right)^2-\frac{1}{21}\frac{R_5}{R_7}\right]^{1/2}$	See Note 8	2, 8
		$1>7>1$	$\frac{1}{7}\frac{R_6}{R_7}-\left[\frac{1}{49}\left(\frac{R_6}{R_7}\right)^2-\frac{1}{21}\frac{R_5}{R_7}\right]^{1/2}$		
	$6+3$	$6>3$	$\frac{1}{7}\frac{R_6}{R_7}+\left[\frac{84}{715}\left(\frac{R_6}{R_7}\right)^2-\frac{4}{35}\frac{R_5}{R_7}\right]^{1/2}$	$\frac{1}{3}(A-6r_p)$	
		$3>6$	$\frac{1}{7}\frac{R_6}{R_7}-\left[\frac{84}{715}\left(\frac{R_6}{R_7}\right)^2-\frac{4}{35}\frac{R_5}{R_7}\right]^{1/2}$		
	$5+4$	$5>4$	$\frac{1}{7}\frac{R_6}{R_7}+\left[\frac{9}{98}\left(\frac{R_6}{R_7}\right)^2-\frac{3}{14}\frac{R_5}{R_7}\right]^{1/2}$	$\frac{1}{4}(A-5r_p)$	
		$4>5$	$\frac{1}{7}\frac{R_6}{R_7}-\left[\frac{9}{98}\left(\frac{R_6}{R_7}\right)^2-\frac{3}{14}\frac{R_5}{R_7}\right]^{1/2}$		
10	$10+0$	-	$A/10$	-	
	$9+1$	Both	$\frac{1}{9}\frac{R_7}{R_8}$	$A-9r_p$	
	$8+2$	$8>2$	$\frac{1}{8}\frac{R_7}{R_8}+\left[\frac{1}{64}\left(\frac{R_7}{R_8}\right)^2-\frac{1}{28}\frac{R_6}{R_8}\right]^{1/2}$	$\frac{1}{2}(A-8r_p)$	
		$2>8$	$\frac{1}{8}\frac{R_7}{R_8}-\left[\frac{1}{64}\left(\frac{R_7}{R_8}\right)^2-\frac{1}{28}\frac{R_6}{R_8}\right]^{1/2}$		
	$8+1+1$	$8>1>1$ and $1>8>1$	$\frac{1}{8}\frac{R_7}{R_8}+\left[\frac{1}{64}\left(\frac{R_7}{R_8}\right)^2-\frac{1}{28}\frac{R_6}{R_8}\right]^{1/2}$	See Note 9	2, 9
		$1>1>8$	$\frac{1}{8}\frac{R_7}{R_8}-\left[\frac{1}{64}\left(\frac{R_7}{R_8}\right)^2-\frac{1}{28}\frac{R_6}{R_8}\right]^{1/2}$		
	$7+3$	$7>3$	$\frac{1}{8}\frac{R_7}{R_8}+\left[\frac{7}{192}\left(\frac{R_7}{R_8}\right)^2-\frac{1}{12}\frac{R_6}{R_8}\right]^{1/2}$	$\frac{1}{3}(A-7r_p)$	

Order	Root Configuration	Root Size Orientation	Primary Root Algorithm	Secondary Root Algorithm	Note
10		$3 > 7$	$\frac{1}{8}\frac{R_7}{R_8} - \left[\frac{7}{192}\left(\frac{R_7}{R_8}\right)^2 - \frac{1}{12}\frac{R_6}{R_8}\right]^{1/2}$		
	$6 + 4$	$6 > 4$	$\frac{1}{8}\frac{R_7}{R_8} + \left[\frac{21}{320}\left(\frac{R_7}{R_8}\right)^2 - \frac{3}{20}\frac{R_6}{R_8}\right]^{1/2}$	$\frac{1}{4}(A - 6r_p)$	
		$4 > 6$	$\frac{1}{8}\frac{R_7}{R_8} - \left[\frac{21}{320}\left(\frac{R_7}{R_8}\right)^2 - \frac{3}{20}\frac{R_6}{R_8}\right]^{1/2}$		
	$5 + 5$	-	$\frac{1}{8}\frac{R_7}{R_8} \pm \left[\frac{7}{64}\left(\frac{R_7}{R_8}\right)^2 - \frac{1}{4}\frac{R_6}{R_8}\right]^{1/2}$	$\frac{1}{5}(A - 5r_p)$	1

Table 2.2 - DDR Multiple Root Algorithms for Orders 3 to 10.

Notes to Table 2.2.

Note 1. - The positive square roots gives the larger of the polynomial roots, the negative square root the smaller.

Note 2. - These algorithms for r_p for an (n-2) + 1 + 1 configuration are the same as the algorithm for an (n-2) + 2 configuration, and consequently, if r_p is verified, for complete verification, the secondary/tertiary roots must also be verified by division into the original equation.

Note 3. - The secondary and tertiary roots for the (n-2) + 1 + 1 configuration are given in the following table.

Note	**Root Configuration**	**Secondary and Tertiary Roots**
3	2 + 1 + 1	$\left\{A - 2r_p \pm \left[A^2 + 4Ar_p - 8r_p^2 - 4B\right]^{1/2}\right\}/2$
4	3 + 1 + 1	$\left\{A - 3r_p \pm \left[A^2 + 6Ar_p - 15r_p^2 - 4B\right]^{1/2}\right\}/2$
5	4 + 1 + 1	$\left\{A - 4r_p \pm \left[A^2 + 8Ar_p - 24r_p^2 - 4B\right]^{1/2}\right\}/2$
6	5 + 1 + 1	$\left\{A - 5r_p \pm \left[A^2 + 10Ar_p - 35r_p^2 - 4B\right]^{1/2}\right\}/2$
7	6 + 1 + 1	$\left\{A - 6r_p \pm \left[A^2 + 12Ar_p - 48r_p^2 - 4B\right]^{1/2}\right\}/2$
8	7 + 1 + 1	$\left\{A - 7r_p \pm \left[A^2 + 14Ar_p - 63r_p^2 - 4B\right]^{1/2}\right\}/2$
9	8 + 1 + 1	$\left\{A - 8r_p \pm \left[A^2 + 16Ar_p - 80r_p^2 - 4B\right]^{1/2}\right\}/2$

Table 2.3 - Secondary and Tertiary Roots for an (n-2)+1+1 Configuration.

Thus it is clear that if a polynomial contains a set of multiple roots as in Table 2.1, subsequent to the determination of the Differential Division Remainder terms, (see below), the roots can very easily and quickly be determined via the algorithms of Tables 2.2 and 2.3.
Note however, the method does not cover those polynomials containing multiple roots together with complex conjugate pairs.

2.3 Generalisation.

The Section above, together with Section 2.3.1 below, gives all the information for determining the multiple roots of polynomials for orders 3 to 10. To generalise the method for polynomials of any order, (3 to n) the following generalised algorithms apply.

2.3.1 The Differential Division Remainder Terms and Ratios.

The Differential Division Remainder Terms and Ratios for any polynomial of order n are given as follows

$$R_{(n-2)} = \frac{2}{n}B - \frac{(n-1)}{n^2}A^2$$

$$R_{(n-3)} = \frac{3}{n}C - \frac{(n-2)}{n^2}AB$$

$$R_{(n-4)} = \frac{4}{n}C - \frac{(n-3)}{n^2}AC \tag{2.11}$$

$$\vdots$$

$$R_1 = \frac{n-1}{n}K_1 - \frac{2}{n^2}AK_2$$

$$R_0 = K_0 - \frac{1}{n^2}AK_1$$

Where K_2, K_1 and K_0 are the final three coefficient terms of the polynomial. The ratios of interest are therefore

$$\frac{R_{(n-3)}}{R_{(n-2)}} = P = \frac{3nC - (n-2)\,AB}{2nB - (n-1)\,A^2}$$

$$\frac{R_{(n-4)}}{R_{(n-2)}} = Q = \frac{4nD - (n-3)\,AC}{2nB - (n-1)\,A^2} \tag{2.12}$$

2.3.2 ***The Primary Root.***

In any polynomial of order n of the type in Table 2.1, if the number of primary roots is L, and the number of secondary roots is S, then it can be shown that

$$\frac{R_{(n-3)}}{R_{(n-2)}} = P = (L-1)\ r_p + (S-1)\ r_s$$

and (2.13)

$$\frac{R_{(n-4)}}{R_{(n-2)}} = Q = 0.5\,(L-2)\,(L-1)\ r_p^2 + (L-1)\,(S-1)\ r_p r_s + (S-2)\ r_s^2$$

From (2.13) the primary root is obtained by solving for r_p thus

$$r_p = \frac{-\left(S^2-4S+5\right)P}{\left(-LS^2+4LS-5L-2S+4\right)} \pm \left[\frac{\left(S^2-4S+5\right)^2 P^2}{\left(-LS^2+4LS-5L-2S+4\right)^2} - \frac{2\,(S-2)\,P^2 - 2\,(S-1)^2\,Q}{\left(-LS^2+4LS-5L-2S+4\right)(L-1)}\right]^{1/2} \tag{2.14}$$

and substitution for P and Q from (2.12) then gives the value of r_p. As stated before, the positive square root in (2.14) gives r_p when $r_p > r_s$ and vice versa for the negative square root. Eq. (2.14) gives r_p for any configuration of L and S, except the $(n+0)$ configuration as is clear from Table 2.2.

2.3.3 ***Secondary and Tertiary Roots.***

The secondary roots are given, as shown above, by

$$r_s = (A - Lr_p)/S \tag{2.15}$$

For an (n-2) + 2 configuration, this gives the correct secondary root, but when the configuration is (n-2) + 1 + 1, (2.16) does not verify by giving a zero remainder when divided into the original equation. This confirms there are two single roots, which can then be determined thus

$$r_s = \frac{\{A - (n-2)\,r_p\}}{2} \pm \frac{\left\{A^2 + (2n-4)\,Ar_p - n\,(n-2)\,r_p^2 - 4B\right\}^{1/2}}{2} \tag{2.16}$$

where the larger of the two single roots is given by the positive square root, and the smaller by the negative square root.
Thus from (2.12), (2.14), (2.15) and (2.16) the multiple roots of any polynomial of any order, with the configuration of Table 2.1, can be determined. Two simple examples of this process are shown in Appendix A.

2.4 Implementation.

2.4.1 ***Manual Implementation.***

If maximum precision is required, it is necessary to perform all calculations manually, i.e. without the use of any mechanical or electronic aid. This is because all such devices can only display a limited number of significant places, and if calculations in which very large numbers, or a large number of decimal places are involved, this will cause minor errors in the results. Manual calculations are not difficult with the equations involved as there are few required and they are all relatively short and simple to perform. This is demonstrated in the examples in Appendix A.

2.4.2 ***Computer Implementation.***

Computer implementation is subject to the same difficulty as mentioned above in the restriction of the number of significant places used in the calculations. In this paper the computer implementation adopted, is the Microsoft EXCEL spreadsheet program via an update of the BAIRSTOW.XLS spreadsheet presented in [1]. EXCEL truncates all numbers to just 15 significant places, and this adversely affects the calculations in two ways.

Firstly, in the determination of the possible roots via the algorithms presented in Table 2.2. This can be partly alleviated by restricting all such calculations to four decimal places. The main problem however, comes in the verification of the possible roots. The rounding errors introduced by EXCEL's 15 significant place restriction makes it extremely difficult to obtain a zero result when dividing the possible roots into the original polynomial. This is because these rounding errors are unpredictable. Over a very large range of numbers, they follow a cubic relationship to the polynomial constant term. However, on top of this there is a short range pseudo-periodic variation, which cannot be precisely mathematically represented. Therefore, to overcome this in the EXCEL presentation here, verification of possible roots by dividing them into the original equation is not used. Instead all the unused DDR remainder terms are utilised. These are all polynomic functions of the primary root and together with an additional check on the polynomial constant term, provide a good verification mechanism.

It is important to note that all the coefficients of the original polynomial must be covered in the verification process of any root, and therefore as necessary all the DDR remainder terms are

included in the EXCEL implementation root verification process. Consequently, for interest, a complete list of all these terms for polynomials of orders 3 to 10 are shown in Appendix B.
Note that because of the anomaly concerning the (n-2) + 2 and (n-2) + 1 + 1 configurations, (see Note 2 to Table 2.2), the latter configuration has been omitted from the EXCEL computer implementation here. This configuration will be included in the method to be described in the next paper for polynomials containing a mixture of multiple and single roots.
Albeit this verification process provides excellent results, it does not completely eliminate EXCEL's rounding errors. There are two types of errors encountered. Firstly, when the polynomial contains multiple roots in the configurations of Table 2.1, and these roots are very close together, the spreadsheet implementation can incorrectly determine them. The points at which errors start to appear, for polynomials of order 10, are shown in the following Table together with the level of errors experienced.

Order	Config'n	Primary Root	Secondary Root	% Root Root Difference	Primary Root Found	Secondary Root Found	Config'n Found	Primary Root % Error	Secondary Root % Error	Note	Test Number
10	9 + 1	13123	13094	0.22	13123.0001	13093.9991	9 + 1	7.0E- 7	6.8E - 6	2	1
		50.1234	50.1050	0.037	50.1233	50.1146	8 + 2	1.9E - 4	1.9E - 4	3	2
		1.9876	1.9875	-	1.9876	1.9875	9 + 1	-	-	1	3
		0.1234	0.1233	-	0.1234	0.1233	9 + 1	-	-	1	4
		0.0015	0.0014	-	0.0015	0.0014	9 + 1	-	-	1	5
	8 + 2	13123	12866	1.96	13122.9998	12866.0008	8 + 2	1.5E - 6	6.2E - 6	2	6
		50.1234	49.7562	0.73	50.1235	49.7558	8 + 2	3.9E - 4	8.0E - 4	2	7
		1.9876	1.9875	-	1.9876	1.9875	8 + 2	-	-	1	8
		0.1234	0.1233	-	0.1234	0.1233	8 + 2	-	-	1	9
		0.0015	0.0014	-	0.0015	0.0014	8 + 2	-	-	1	10
	7 + 3	13123	12853	2.06	13123.0001	12852.9998	7 + 3	7.0E - 7	1.5E - 6	2	11
		50.1234	49.6989	0.85	50.1233	49.6991	7 + 3	1.9E - 4	4.0E - 4	2	12
		1.9876	1.9848	0.14	1.9874	1.9842	8 + 2	1.0E - 2	3.0E - 2	3	13
		0.1234	0.1233	-	0.1234	0.1233	7 + 3	-	-	1	14
		0.0015	0.0014	-	0.0015	0.0014	7 + 3	-	-	1	15
	6 + 4	13123	12838	2.17	13122.9999	12838.0002	6 + 4	7.0E - 7	1.5E - 6	2	16
		50.1234	49.9802	0.29	50.1238	49.9796	6 + 4	7.9E - 4	1.2E - 3	2	17
		1.9876	1.9851	0.13	1.9872	1.9842	8 + 2	2.0E - 2	4.5E - 2	3	18
		0.1234	0.1233	-	0.1234	0.1233	6 + 4	-	-	1	19
		0.0015	0.0014	-	0.0015	0.0014	6 + 4	-	-	1	20

Order	Config'n	Primary Root	Secondary Root	% Root Root Difference	Primary Root Found	Secondary Root Found	Config'n Found	Primary Root % Error	Secondary Root % Error	Note	Test Number
10	5 + 5	13123	12846	2.11	13123.0001	12845.9999	5 + 5	7.0E - 7	7.0E - 7	2	21
		50.1234	49.9621	0.32	50.1244	49.9611	5 + 5	2.0E - 3	2.0E - 3	2	22
		1.9876	1.9844	0.16	**Not Recognised**			**No Result**		4	23
		0.1234	0.1233	-	0.1234	0.1233	5 + 5	-	-	1	24
		0.0015	0.0014	-	0.0015	0.0014	5 + 5	-	-	1	25

Table 2.4 - Test Results, the Point of Onset of Errors.

Notes to Table 2.4.

1 These tests gave the correct root configuration and value with the primary and secondary roots being as close as possible, (12 of 25).

2 These tests gave the correct root configuration but with small errors in the primary and secondary root values, (9 of 25).

3 These tests gave the incorrect root configuration plus a small error in the values of the primary and secondary roots, (3 of 25).

4 In this test the multiple root configuration was not recognised by the verification process, (and analysis of the polynomial would thereby be passed to the BAIRSTOW Module in the spreadsheet), (1 of 25).

To illustrate the vagaries of the Excel rounding error, with respect to Note 4 for test #23, the root combination of 1.9876 x 5 plus 1.9843 x 5 was correctly recognised for configuration and values, as was the combination 1.9876 x 5 plus 1.9846 x 5, in contrast to the actual test result above in which the secondary root lies in between the above two values.
The second type of errors occur when analysing polynomials containing singular roots, with or without some multiple configuration. Again, when the roots are very close together, they can be incorrectly recognised as one of the multiple configurations of Table 2.1. The point at which this occurs, for polynomials of order 10 are shown in Table 2.5 below.

Primary Root Size	**Root Difference**
r_p >1000	0.1
$1000 > r_p > 100$	0.01
$100 > r_p > 10$	0.001
r_p <10	0.0001

Table 2.5 - Closeness of Multiple/Singular Root Combinations Generating Multiple Root Configuration and Value Errors.

While the above error regime of this process has been fairly extensively described above, it is clear that these errors are extremely small, and for most purposes, can be ignored. However, one caveat that must be mentioned, is that all of the tests reported here are spot checks only. It is impossible to test all possible multiple root configuration values as they are infinite in number. Consequently the error regime tables above must be treated as guidelines only and when roots found are closer than those in Tables 2.4 and 2.5, the error regime may change.

The error regime for orders 3 to 9 are similar or less than those shown above for order 10.

A comparison of the accuracy of this process compared to the BAIRSTOW module, for a representative multiple root configuration, (roots adequately separated), is shown as part of the examples of Appendix A.
The EXCEL spreadsheet in which this process has been incorporated, BAIRSTOW2.XLS can be downloaded here.

www.relativitydomains.com/Mathematics/MultipleDDR/BAIRSTOW2.ZIP

This ZIP file also includes a slightly updated Polynomial Construction spreadsheet, (POLYNOMIALCONSTRUCTION1.1.0.XLS).

3 Conclusions.

The Differential Division Remainder Method described here provides a very simple and quick way to determine whether any polynomial of any order contains combinations of multiple root pairs, and if so, their values.
Precise determination requires manual computation, but even with the rounding errors inherent in EXCEL, the results achieved with the DDR module, are adequately accurate for most purposes. Where the roots are very close, the maximum errors in their determination, for a large range of 10^{th} order polynomials, are less that 0.05%. However, as stated previously, this error regime is based solely upon a series of spot checks and must therefore be treated as a guideline only. Consequently, it is recommended that if this process provides roots that are closer than those shown in Table 2.4 and 2.5, they should be used to construct the applicable polynomial for comparison with the original equation, and as necessary and desired, amended using the iteration process of the Polynomial Construction Spreadsheet as described in [1]. This is particularly so if the coefficients of the original polynomial are all integers, and the roots found are not.
The DDR Method is applicable to 39 multiple root configurations for polynomials of orders 3 to 10 as shown in Table 2.1. This however, leaves a further 87 configurations that the method does not cover. These configurations contain a number of multiple roots plus a number of single roots. They will form the subject of the next paper which will introduce a computer method, again utilising EXCEL, to provide precise solutions.

APPENDIX A.

Examples of the Manual Application of the DDR Method.

Consider the equation

$$y = x^{11} + 68x^{10} + 2092x^9 + 38424x^8 + 468006x^7 + 3967824x^6 + 23885148x^5$$
$$+534052104x^4 + 127181473x^3 + 596245132x^2 + 615354464x + 368947264 \tag{A.1}$$

The first four coefficients required to generate the first three remainder terms are A= 68, B= 2092, C= 38424 and D= 468006. Substitution of these coefficients into (2.12), together with the polynomial order gives the DDR remainder ratios as

$$\frac{R_9}{R_{10}} = P = 57$$
$$\frac{R_8}{R_{10}} = Q = 1437 \qquad (A.2)$$

Eq.(A.1) could contain multiple roots in the following configurations

$$(i) L = 10, S = 1$$
$$(ii) L = 9, S = 2 \quad or \quad 1 + 1$$
$$(iii) L = 8, S = 3 \qquad (A.3)$$
$$(iv) L = 7, S = 4$$
$$(v) L = 6, S = 5$$

It would be necessary to determine possible r_p's for all of these combinations and check by dividing them into (A.1) for a zero remainder. Pre-empting the result, the algorithm for r_p for the combination (A.3)(iii) is calculated from (2.14) as

$$r_p = \frac{P}{9} + \left\{ \frac{144P^2}{5103} - \frac{4Q}{63} \right\}^{1/2} \qquad (A.4)$$

Substituting for P and Q from (A.2) then gives

$$r_p = 7.0000 \qquad (A.5)$$

Dividing $(r_p + 7)$ into (A.1) gives a zero remainder confirming that this value contributes 8 roots. The secondary root is then simply

$$r_s = \frac{1}{3}(A - 8r_p)$$
$$= 4.0000 \qquad (A.6)$$

Now consider the equation

$$y = x^{11} + 69x^{10} + 2150x^9 + 39900x^8 + 489510x^7 + 4163334x^6 + 25008816x^5$$
$$+105884100x^4 + 308828625x^3 + 58883324x^2 + 65718314x + 322828856 \qquad (A.7)$$

Here $A = 69$, $B = 2150$, $C = 39900$ and $D = 489510$, which gives

$$\frac{R_9}{R_{10}} = P = 59.5161$$
$$\frac{R_8}{R_{10}} = Q = 1568.9032 \tag{A.8}$$

The algorithm for (A.3)(ii) is

$$r_p = \frac{P}{9} + \left\{ \frac{P^2}{81} - \frac{Q}{36} \right\}^{1/2} \tag{A.9}$$

Substitution of (A.8) into (A.9) then gives

$$r_p = 7.0000 \tag{A.10}$$

Which is confirmed as a root of (A.7). Accordingly if the configuration was $9r_p + 2r_s$ the secondary root would be

$$r_s = \frac{1}{2}(A - 9r_p)$$
$$= 3.0000 \tag{A.11}$$

But this does not give a zero remainder when divided into (A.7) which confirms that the configuration must be $9r_p + r_{s1} + r_{s2}$. From (2.16) the secondary roots are therefore calculated as

$$r_s = \frac{69 - 9x7}{2} \pm \frac{\left\{ 69^2 + 18x69x7 - 11x9x7^2 - 4x2150 \right\}^{1/2}}{2}$$
$$= 3.0000 \pm 1.0000 \tag{A.12}$$
$$= 4.0000 \quad and \quad 2.0000$$

These are confirmed as the secondary roots of (A.7) in the usual manner.
Finally, as an example of the accuracy of the DDR method compared to Bairstow for this category of equation, consider the 7^{th} order polynomial

$$y = (x + 37)^4 (x + 23)^3 \tag{A.13}$$

Expanded this is

$$y = x^7 + 217x^6 + 20130x^5 + 1016421x^4 + 30690723x^3$$
$$+ 550802091x^2 + 5439473711x + 22802916887 \tag{A.14}$$

Using the BAIRSTOW module, this is solved as

$$r_1 = 22.9993$$

$$r_2 = 37.0138$$

$$r_3 = 23.0004 + j0.0006 \tag{A.15}$$

$$r_4 = 23.0004 - j0.0006$$

$$r_{5\,to\,7} = 36.9954$$

The time taken was 170.60 seconds, (400MHz Computer).

Using DDR the solution was exactly as (A.13) and the time taken was 1.10 seconds

Appendix B.

This Appendix presents a complete listing of all the DDR remainder terms for polynomials of orders 3 to 10. These are used in the root verification process in the DDR module for multiple root determination of polynomials with root configurations as in Table 2.1. It is clear from these terms that they exhibit a pattern closely related to Pascal's Triangle.

n	Config'n	$R_0/R_{(n-2)}$	$R_1/R_{(n-2)}$	$R_2/R_{(n-2)}$	$R_3/R_{(n-2)}$	$R_4/R_{(n-2)}$
3	2 + 1	r_p				
4	3 + 1	r_p^2	$2r_p$			
	2 + 2	r_pr_s	$r_p + r_s$			
5	4 + 1	r_p^3	$3r_p^2$	$3r_p$		
	3 + 2	$r_p^2r_s$	$r_p\,(r_p + 2r_s)$	$2r_p + r_s$		
6	5 + 1	r_p^4	$4r_p^3$	$6r_p^2$	$4r_p$	
	4 + 2	$r_p^3r_s$	$r_p^2\,(r_p + 3r_s)$	$3r_p\,(r_p + r_s)$	$3r_p + r_s$	
	3 + 3	$r_p^2r_s^2$	$2r_pr_s\,(r_p + r_s)$	$r_p^2 + 4r_pr_s + r_s^2$	$2(r_p + r_s)$	
7	6 + 1	r_p^5	$5r_p^4$	$10r_p^3$	$10r_p^2$	$5r_p$
	5 + 2	$r_p^4r_s$	$r_p^3\,(r_p + 4r_s)$	$r_p^2\,(4r_p + 6r_s)$	$r_p\,(6r_p + 4r_s)$	$4r_p + r_s$
	4 + 3	$r_p^3r_s^2$	$r_p^2r_s\,(2r_p + 3r_s)$	$r_p\left(r_p^2 + 6r_pr_s + 3r_s^2\right)$	$3r_p^2 + 6r_pr_s + r_s^2$	$3r_p + 2r_s$
8	7 + 1	r_p^6	$6r_p^5$	$15r_p^4$	$20r_p^3$	$15r_p^2$
	6 + 2	$r_p^5r_s$	$r_p^4\,(r_p + 5r_s)$	$5r_p^3\,(r_p + 2r_s)$	$10r_p^2\,(r_p + r_s)$	$5r_p\,(2r_p + r_s)$
	5 + 3	$r_p^4r_s^2$	$2r_p^3r_s\,(r_p + 2r_s)$	$r_p^2\left(r_p^2 + 8r_pr_s + 6r_s^2\right)$	$4r_p\left(r_p^2 + 3r_pr_s + r_s^2\right)$	$6r_p^2 + 8r_pr_s + r_s^2$
	4 + 4	$r_p^3r_s^3$	$3r_p^2r_s^2\,(r_p + r_s)$	$3r_pr_s\left(r_p^2 + 3r_pr_s + r_s^2\right)$	$r_p^3 + 9r_p^2r_s + 9r_pr_s^2 + r_s^3$	$3r_p^2 + 9r_pr_s + 3r_s^2$
9	8 + 1	r_p^7	$7r_p^6$	$21r_p^5$	$35r_p^4$	$35r_p^3$
	7 + 2	$r_p^6r_s$	$r_p^5\,(r_p + 6r_s)$	$r_p^4\,(6r_p + 15r_s)$	$5r_p^3\,(r_p + 4r_s)$	$5r_p^2\,(4r_p + 3r_s)$
	6 + 3	$r_p^5r_s^2$	$r_p^4r_s\,(2r_p + 5r_s)$	$r_p^3\left(r_p^2 + 10r_pr_s + 10r_s^2\right)$	$5r_p^2\left(r_p^2 + 4r_pr_s + 2r_s^2\right)$	$5r_p\left(2r_p^2 + 4r_pr_s + r_s^2\right)$
	5 + 4	$r_p^4r_s^3$	$r_p^3r_s^2\,(3r_p + 4r_s)$	$3r_p^2r_s\left(r_p^2 + 4r_pr_s + 2r_s^2\right)$	$r_p\left(r_p^3 + 12r_p^2r_s + 18r_pr_s^2 + 4r_s^3\right)$	$4r_p^3 + 18r_p^2r_s + 12r_pr_s^2 + r_s^3$
10	9 + 1	r_p^8	$8r_p^7$	$28r_p^6$	$56r_p^5$	$70r_p^4$
	8 + 2	$r_p^7r_s$	$r_p^6\,(r_p + 7r_s)$	$r_p^5\,(7r_p + 21r_s)$	$7r_p^4\,(3r_p + 5r_s)$	$35r_p^3\,(r_p + r_s)$
	7 + 3	$r_p^6r_s^2$	$r_p^5r_s\,(2r_p + 6r_s)$	$r_p^4\left(r_p^2 + 12r_pr_s + 15r_s^2\right)$	$r_p^3\left(6r_p^2 + 30r_pr_s + 20r_s^2\right)$	$5r_p^2\left(3r_p^2 + 8r_pr_s + 3r_s^2\right)$

n	Config'n	$R_0/R_{(n-2)}$	$R_1/R_{(n-2)}$	$R_2/R_{(n-2)}$	$R_3/R_{(n-2)}$	$R_4/R_{(n-2)}$
10	6 + 4	$r_p^5 r_s^3$	$r_p^4 r_s^2 (3r_p + 5r_s)$	$r_p^3 r_s \left(3r_p^2 + 15r_p r_s + 10r_s^2\right)$	$r_p^2 \left(r_p^3 + 15r_p^2 r_s + 80r_p r_s^2 + 10r_s^3\right)$	$5r_p \left(r_p^3 + 6r_p^2 r_s + 6r_p r_s^2 + r_s^3\right)$
	5 + 5	$r_p^4 r_s^4$	$r_p^3 r_s^3 (4r_p + 4r_s)$	$2r_p^2 r_s^2 \left(3r_p^2 + 8r_p r_s + 3r_s^2\right)$	$4r_p r_s \left(r_p^3 + 6r_p^2 r_s + 6r_p r_s^2 + r_s^3\right)$	$r_p^4 + 16r_p^3 r_s + 36r_p^2 r_s^2 + 16r_p r_s^3 + r_s^4$

n	Config'n	$R_5/R_{(n-2)}$	$R_6/R_{(n-2)}$	$R_7/R_{(n-2)}$
8	7 + 1	$6r_p$		
	6 + 2	$5r_p + r_s$		
	5 + 3	$4r_p + 2r_s$		
	4 + 4	$3r_p + 3r_s$		
9	8 + 1	$21r_p^2$	$7r_p$	
	7 + 2	$r_p (15r_p + 6r_s)$	$6r_p + r_s$	
	6 + 3	$10r_p^2 + 10r_p r_s + r_s^2$	$5r_p + 2r_s$	
	5 + 4	$6r_p^2 + 12r_p r_s + 3r_s^2$	$4r_p + 3r_s$	
10	9 + 1	$56r_p^3$	$28r_p^2$	$8r_p$
	8 + 2	$7r_p^2 (5r_p + 3r_s)$	$r_p (21r_p + 7r_s)$	$7r_p + r_s$
	7 + 3	$r_p \left(20r_p^2 + 30r_p r_s + 6r_s^2\right)$	$15r_p^2 + 12r_p r_s + r_s^2$	$6r_p + 2r_s$
	6 + 4	$10r_p^3 + 30r_p^2 r_s + 15r_p r_s^2 + r_s^3$	$10r_p^2 + 15r_p r_s + 3r_s^2$	$5r_p + 3r_s$
	5 + 5	$4r_p^3 + 24r_p^2 r_s + 24r_p r_s^2 + 4r_s^3$	$6r_p^2 + 16r_p r_s + 6r_s^2$	$4r_p + 4r_s$

Table B.1 - Summary of All DDR Remainder Term Ratios for Polynomials of Order 3 to 10.

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