THE SERIES SUM OF DIVERGENT

ALTERNATING INFINITE SERIES

AND

THE NATURE OF INFINITY.

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ABSTRACT.

This paper presents three methods for determining the series sums of divergent alternating infinite series. The results subsequently enable a discussion on the nature of infinity and its relationship with the finite numbers.

Contents.

1.0 Introduction.

In [1] the closed forms of the infinite series resulting from the integration of Newton's expansion of *ln*(1 + *x*) was explored. This showed that there was an unlimited number of infinite series for which the closed form could be obtained. In that analysis the independent variable, *x*, was restricted to unity.

In this paper the same function, *ln*(1 + *x*), is explored for infinite series in the opposite analytical direction, differentiation. Contrary to the analysis in [1], here the independent variable, *x*, can take values greater than unity. The infinite series so derived all exhibit alternating term values and are currently classified as divergent. However, it will be seen that they all possess a finite series sum as the term number becomes infinite.

In [2], a method was presented to estimate the series sum of non-closed form convergent alternating infinite series to any desired degree of precision. This paper presents an alternative method requiring less computation to achieve the same result.

Finally, the results obtained enable a discussion on the nature of infinity, which provides some insight into its relationship with the finite numbers.

2.0 The Series Sum of Divergent Alternating Infinite Series.

2.1 Preamble and Derivation of a Generalisation.

It is well known that when a convergent infinite series is differentiated with respect to its independent variable, its degree of convergence weakens and the resulting series may even diverge. However, if the original and therefore subsequent series consists of terms alternating in sign, they will possess a finite series sum. The series chosen to explore this attribute is Newton's expansion of *ln*(1 + *x*), thus

$$\boldsymbol{ln}(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\cdots \tag{2.1}$$

If (2.1) is differentiated *m* times, it can be shown to result in the following generalisation.

$$\frac{1}{(1+x)^m}=\frac{1}{(m-1)!}\sum_{t=1}^{\infty}(-1)^{(t-1)}\frac{(m+t-2)!}{(t-1)!}x^{(t-1)} \tag{2.2}$$

where clearly *t* is the term number.

It would appear that the term on the left of (2.2) is the closed form of the series. However, even with *x* and *m* at their lowest value, unity, (2.2) reduces to

$$\frac{1}{2}=1-1+1-1+1-\cdots \tag{2.3}$$

and clearly the partial sums of (2.3) continuously alternate between +1 and 0 and are independent of term number *t*. It is generally accepted that the series sum of the right hand side of (2.3) is 1/2, as the left hand side indicates.

Carrying this interpretation forward to all values of *x* and *m*, then the left hand side cannot be regarded as a true closed form, but instead a series sum which will be shown to be the average of adjacent partial sum values as $t \to \infty$. Proof of the validity of this interpretation is necessary and is effected in the following Sections.

2.2 Series With $x = 1$ and $m > 1$.

In (2.2) insert $x = 1$ and any value greater than unity for m. Now calculate $m + 4$ partial sums of the series. Plot these and determine the equations of the envelope curves in terms of the term number. The sum of these two curves will be a pure rational number independent of term number and, half of this will be the series sum. Because the result is independent of term number, it will apply as $t \to \infty$ and this therefore is a sufficient proof of the interpretation for this scenario. Examples of this process are shown in Section 4.0 for $m = 2$ and 3.

2.3 Series With $x > 1$ and $m = 1$.

With $x > 1$, the method of proof of the previous Section is not practical because determination of the envelope curve equations is not simple. However, it is well known that the series sum of these series is given by

$$S = \frac{1}{1+x} \tag{2.4}$$

as they are of geometric form. However, the method presented here is required for those series for when x and m are both > 1 as described in Section 2.4 below. The method of proof here is based upon a unique relationship between the terms of the partial sums.

Determine any number of partial sums desired, (≥ 6). Take any three adjacent positive or negative values from this sequence. Calculate the difference between each adjacent term as D1 and D2, and then the difference between D1 and D2 as D3. The series sum is then given by

$$S = \frac{D1}{D3}(x-1) = \frac{1}{x+1} \tag{2.5}$$

Because this calculation can be conducted with any three adjacent positive or negative sums, including those as $t \to \infty$, this is a sufficient proof of the interpretation for this scenario. In addition, the unique relationship between the terms of the partial sums is shown as follows.

If any three adjacent partial sums are

$$Top\ P.S.\ Sequence = x^{(n-1)} - x^{(n-2)} + \cdots + 1,\ x^{(n+1)} - x^{n} + \cdots + 1,\ x^{(n+3)} - x^{(n+2)} + \cdots + 1 \tag{2.6}$$

then D1, D2 and D3 are given by

$$\begin{aligned} D1 &= x^{(n+1)} - x^{n} \\ D2 &= x^{(n+3)} - x^{(n+2)} \\ D3 &= x^{(n+3)} - x^{(n+2)} - x^{(n+1)} + x^{n} \end{aligned} \tag{2.7}$$

so that

$$\frac{D1}{D3} = \frac{x^{(n+1)} - x^{n}}{x^{(n+3)} - x^{(n+2)} - x^{(n+1)} + x^{n}} \tag{2.8}$$

and this reduces to

$$\frac{D1}{D3} = \frac{x-1}{x^3 - x^2 - x + 1} \tag{2.9}$$

and therefore

$$\frac{D1}{D3}(x-1)=\frac{(x-1)^2}{x^3-x^2-x+1}=\frac{1}{x+1} \tag{2.10}$$

This confirms the method of proof.

2.4 Series With *x* > 1 and *m* > 1.

Proof for this family of series is simply

$$S=\left\{\frac{D1}{D3}(x-1)\right\}^m \tag{2.11}$$

This is so because the series derived from (2.2) will be

$$\frac{1}{(1+x)^m}=1-mx+\frac{m(m+1)\,x^2}{2!}-\frac{m(m+1)(m+2)\,x^3}{3!}+\cdots \tag{2.12}$$

and this is easily shown to be

$$\frac{1}{(1+x)^m}=\left(1-x+x^2-x^3+\cdots\right)^m \tag{2.13}$$

Examples for this scenario are given below in Section 4.0 for $x = 2$ with $m = 6$.

3.0 A Partial Sum Iterative Method for Convergent Infinite Series Sum Estimation.

In [2] for those series that did not have a closed form, a method of estimating the series sum to a desired precision was presented. Here an alternative method is shown that is somewhat quicker than that in [2]. This method is particularly applicable to series of the form

$$S(t)=\sum_{t=1}^{\infty}(-1)^{(t-1)}\frac{1}{(xt-x+1)^m} \tag{3.1}$$

The series sum is given by the average of a sequence of recursive partial sums until the desired precision is reached. The method is best illustrated by an example and one for $x = 1$ and $m = 3$, to a precision of 1E-8, together with several others, is given in Section 4.0 below.

4.0 Examples From Sections 2.0 and 3.0.

4.1 Examples From Section 2.2.

(i) From (2.2) derive the series for $m = 2$ as

$$\frac{1}{(1+x)^2}=1-2x+3x-4x+5x-\cdots \tag{4.1}$$

Put $x = 1$ to give

$$\frac{1}{4}=1-2+3-4+5-\cdots \tag{4.2}$$

Determine the sequence of 6 partial sums from the right hand side to give

$$P.S. = -1, +2, -2, +3, -3, +4 \tag{4.4}$$

Plot these plus the envelope curves, thus

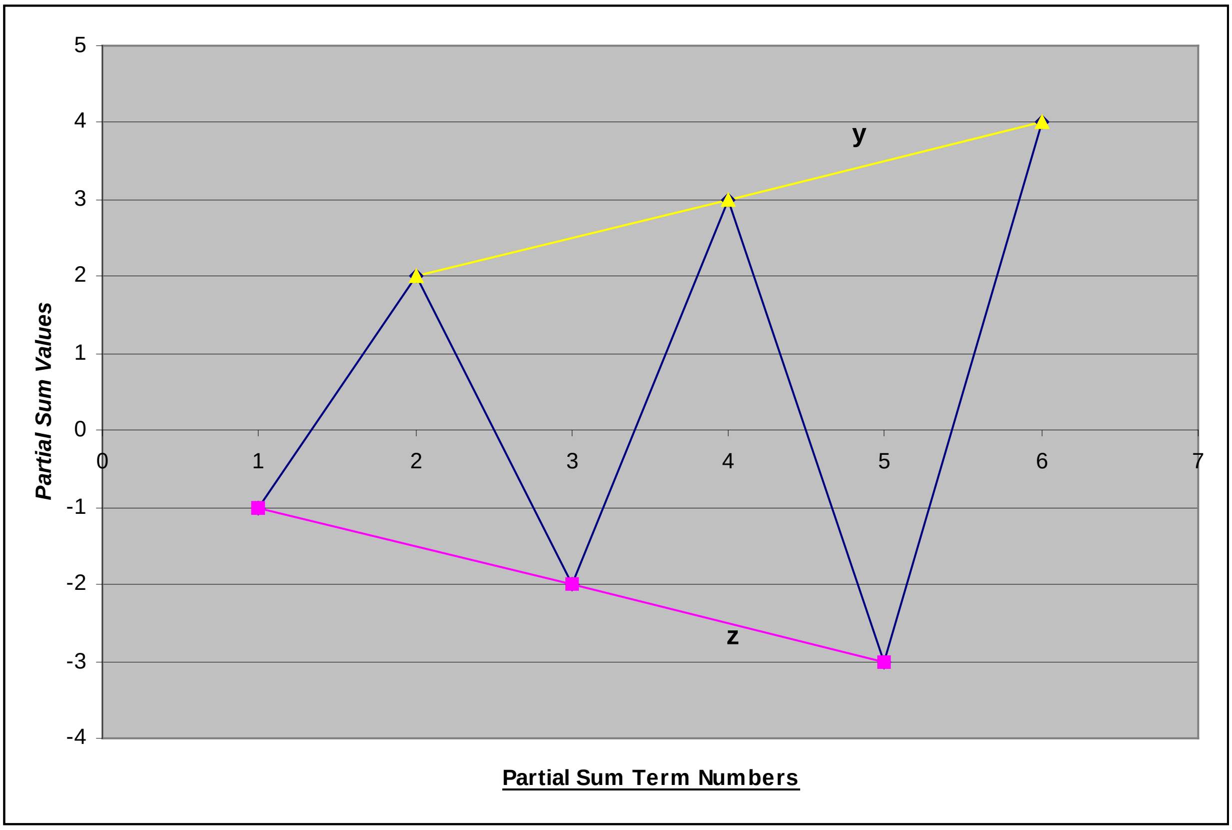

Fig. 4.1 - Partial Sums of Eq. 4.2.

The locus of each envelope curve is clearly linear and therefore given by the following simple equations

$$\begin{aligned} y &= at + b \\ z &= ct + d \end{aligned} \tag{4.3}$$

Via simple simultaneous equation analysis, the equation of each curve is

$$\begin{aligned} y &= \frac{t}{2} + 1 \\ z &= -\frac{t}{2} - \frac{1}{2} \end{aligned} \tag{4.4}$$

So that the average of these curves is then clearly independent of *t* and is

$$\frac{y+z}{2} = \frac{1}{4} \tag{4.5}$$

which concurs with (4.2),

(ii) When m = 3, (2.2) yields

$$\frac{1}{(1+x)^3} = 1 - 3x + 6x^2 - 10x^3 + 15x^4 - 21x^5 + \cdots \tag{4.6}$$

Putting x = 1, calculating seven partial sums and plotting these plus their envelope curves shows that they are quadratic and therefore given by

$$\begin{aligned} y &= at^2 + bt + c \\ z &= dt^2 + et + f \end{aligned} \tag{4.7}$$

Simultaneous equation analysis then yields

$$\begin{aligned} y &= \frac{t^2}{4} + t + 1 \\ z &= -\frac{t^2}{4} - t - \frac{3}{4} \end{aligned} \tag{4.8}$$

So that

$$\frac{y+z}{2} = \frac{1}{8} \tag{4.9}$$

which concurs with the left hand side of (4.6) when x = 1.
This process can be repeated for any value of m and will always yield polynomial envelope equations of order (m - 1). Their average is always independent of partial sum sequence number, therefore giving a series sum equal to $1/2^m$ and thus confirming the proof of Section 2.2. The method can be used for any alternating infinite series for which the equations of the partial sum envelope curves can be determined.

4.2 Examples From Sections 2.3 and 2.4.

Repeating (2.12)

$$\frac{1}{(1+x)^m} = 1 - mx + \frac{m(m+1)\,x^2}{2!} - \frac{m(m+1)(m+2)\,x^3}{3!} + \cdots \tag{4.10}$$

First put x = 2 and m = 1 to give

$$\frac{1}{3} = 1 - 2 + 4 - 8 + 16 - 32 + 64 - \cdots \tag{4.11}$$

Calculate any sequence of six partial sums and take any three adjacent positive or negative values, thus

$$P.S. = -1, +3, -5, +11, -21, +43 \tag{4.12}$$

taking the three positive values, 3, 11 and 43, then

$$\begin{aligned} D1 &= 11 - 3 = 8 \\ D2 &= 43 - 11 = 32 \\ D3 &= 32 - 8 = 24 \end{aligned} \tag{4.13}$$

To give

$$S(1) = \frac{D1}{D3}(x-1) = \frac{1}{3} \tag{4.14}$$

As per (4.11).

Now in (4.10) put m = 6, (and x = 2), to give the series

$$S(6) = 1 - 12 + 84 - 448 + \cdots \tag{4.15}$$

So that from(2.9) and (4.14)

$$S(6) = \left(\frac{1}{3}\right)^6 = \frac{1}{729} \tag{4.16}$$

4.3 Examples From Section 3.0.

Here the examples demonstrating the method in [2] are repeated to effect a comparison.
First consider example (i) of Appendix A of [2]. To determine η(3) to a precision of 1E-8, a total of 49 partial sums were needed. In the method of this paper Section 3.0, only 18 initial partial sums need to be determined to achieve the same level of precision, (repeat of 8^{th} decimal place). The sequence of calculations that arrive at that result are shown below, (6 columns only shown).The final result to the precision stated is at the bottom of Term Number1 column.

Actual Value, (15 Places)	9.015426773696880E-01					
Term Number	1	2	3	4	5	6
Series Terms	1.0	-0.125	0.037037037	-0.015625	0.008	-0.00462963
Partial Sums	8.7500000000000000E-01	9.1203703703703700E-01	8.9641203703703700E-01	9.0441203703703700E-01	8.9978240740740700E-01	9.0269785930245100E-01
Average 1	8.9351851851851800E-01	9.0422453703703700E-01	9.0041203703703700E-01	9.0209722222222200E-01	9.0124013335492900E-01	9.0172129680245100E-01
Average 2	8.9887152777777800E-01	9.0231828703703700E-01	9.0125462962963000E-01	9.0166867778857600E-01	9.0148071507869000E-01	9.0157595108057200E-01
Average 3	9.0059490740740700E-01	9.0178645833333300E-01	9.0146165370910300E-01	9.0157469643363300E-01	9.0152833307963100E-01	9.0154974598369300E-01
Average 4	9.0119068287037000E-01	9.0162405602121800E-01	9.0151817507136800E-01	9.0155151475663200E-01	9.0153903953166200E-01	9.0154433449233900E-01
Average 5	9.0140736944579400E-01	9.0157111554629300E-01	9.0153484491400000E-01	9.0154527714414700E-01	9.0154168701200100E-01	9.0154309695952200E-01
Average 6	9.0148924249604300E-01	9.0155298023014600E-01	9.0154006102907300E-01	9.0154348207807400E-01	9.0154239198576100E-01	9.0154279041559300E-01
Average 7	9.0152111136309500E-01	9.0154652062961000E-01	9.0154177155357300E-01	9.0154293703191700E-01	9.0154259120067700E-01	9.0154270942703300E-01
Average 8	9.0153381599635200E-01	9.0154414609159200E-01	9.0154235429274500E-01	9.0154276411629700E-01	9.0154265031385500E-01	9.0154268686033700E-01
Average 9	9.0153898104397200E-01	9.0154325019216900E-01	9.0154255920452100E-01	9.0154270721507600E-01	9.0154266858709600E-01	9.0154268028455500E-01
Average 10	9.0154111561807000E-01	9.0154290469834500E-01	9.0154263320979900E-01	9.0154268790108600E-01	9.0154267443582500E-01	9.0154267829381300E-01
Average 11	9.0154201015820800E-01	9.0154276895407200E-01	9.0154266055544200E-01	9.0154268116845600E-01	9.0154267636481900E-01	9.0154267767091300E-01
Average 12	9.0154238955614000E-01	9.0154271475475700E-01	9.0154267086194900E-01	9.0154267876663700E-01	9.0154267701786600E-01	9.0154267747030700E-01
Average 13	9.0154255215544800E-01	9.0154269280835300E-01	9.0154267481429300E-01	9.0154267789225200E-01	9.0154267724408700E-01	9.0154267740404000E-01
Average 14	9.0154262248190100E-01	9.0154268381132300E-01	9.0154267635327200E-01	9.0154267756816900E-01	9.0154267732406300E-01	9.0154267738164900E-01
Average 15	9.0154265314661200E-01	9.0154268008229800E-01	9.0154267696072100E-01	9.0154267744611600E-01	9.0154267735285600E-01	9.0154267737393000E-01
Average 16	9.0154266661445500E-01	9.0154267852150900E-01	9.0154267720341900E-01	9.0154267739948600E-01	9.0154267736339300E-01	9.0154267737121900E-01
Average 17	9.0154267256798200E-01	9.0154267786246400E-01	9.0154267730145200E-01	9.0154267738144000E-01	9.0154267736730600E-01	9.0154267737025200E-01

Table 4.1 - Sequence of Recursive Partial Sum Averages for η(3).

The comparative results for the other examples in [2] are shown in the following table.

Series	Number of Initial Partial Sums Required		Precision
	[2] Appendix A	This Method	
$\eta(3)$	49	17 - 19	1E-8
$\xi(4)$	37	17 - 19	1E-8
$S = 1 - \frac{1}{5^5} + \frac{1}{9^5} - \frac{1}{13^5} + \cdots$	25	12 - 14	1E-8
$S = 1 - \frac{1}{6^6} + \frac{1}{11^6} - \frac{1}{16^6} + \cdots$	61	23 - 25	1E-12
$\eta(1)$	Not Included	36 - 38	1E-15

Table 4.2 - Partial Sums Required to Achieve a Required Precision, [2] Compared to Recursive Average Partial Sum Method.

This method can also be used with series containing an irregular series of terms. However, the rate of convergence is much slower. For instance, for the alternating series of reciprocal primes, it requires an initial partial sum sequence of 21 terms to achieve a precision of three significant figures.

When using this method it is advisable to continue the analysis until the decimal place of interest repeats at least three times in succession, as indicated in the above table, to ensure the precision required has been reached.

In general the method can be used for any alternating series in which the term values reduce. However, for alternating series with increasing term values, when the ratio of the magnitude of adjacent term values is greater than two, the method fails. For ratios < 2 and >1, the method is successful but, convergence is slow and precision is limited.

5.0 The Nature of Infinity.

In the eighteenth century, great mathematicians such as Leonhard Euler, considered infinity to be a number, and indeed Euler himself wrote of finding the minimum value of infinity. Today that idea has been discounted and infinity is generally considered to be the result of a mathematical operation or function that increases or decreases without limit. While that is certainly true, it does not represent all attributes possessed by this concept. To illustrate this consider the following three divergent series.

$$\begin{aligned} S_1 &= 1 + 2 + 3 + 4 + 5 + \cdots \\ S_2 &= 1 + 1 + 1 + 1 + 1 + \cdots \\ S_3 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots \end{aligned} \tag{5.1}$$

If the partial sums of these three series are plotted, the result is as shown in the following figure.

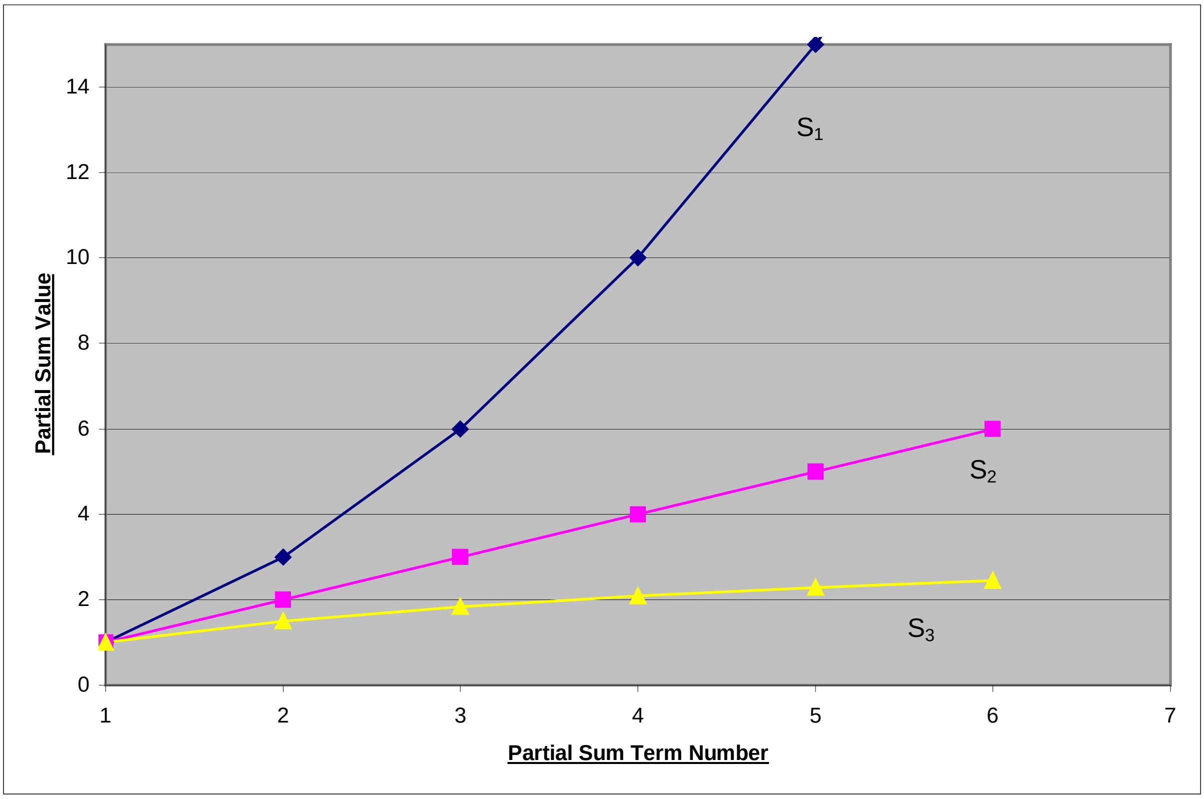

Fig. 5.1 - Partial Sum plot of Divergent Series S_1, S_2 and S_3.

From Fig. 5.1 it is clear that not only are these three series diverging without limit, but they are also very rapidly diverging from one another. This suggests that the degree of infinity to which each series is diverging is different. To further support this possibility, now consider the convergent Eta series

$$\eta(1) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots \tag{5.2}$$

Because only the series sum is being considered here, it is permitted to re-arrange the series thus

$$\eta(1) = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots\right) \tag{5.3}$$

Both the positive and negative series in (5.3) diverge, yet the series sum is *ln*(2). This again suggests that the degree of infinity of the two halves is different. Moreover it also suggests that these two degrees are commutable and their sum is the finite number stated.
As a consequence of the above, all the series of Section 2.0 and their examples in Sections 4.1 and 4.2 exhibit similar characteristics, and it is therefore proposed that in each series the positive and negative terms each diverge to a different level of infinity, and these infinities are commutable to the series sums derived.

The following questions then arise as to

(i) How many degrees of infinity are there?

(ii) What is the largest degree of infinity?

(iii) What is the smallest degree of infinity?

(iv) What sense then should be made of the word "infinity" and the symbol ∞.

On the first three of these questions, definitive answers are possibly not available without some further considerable investigation of infinite series in general. However, tentative answers are proposed as follows.

(i) Because there must be an unlimited number of series of the form of (2.2), it is proposed that there must be an unlimited number of infinities.

(ii) The envelope curves of all of the series discussed here only diverge to an infinity after an infinite number of terms, (as also would all series with all positive terms). It would therefore be expected that the largest degree of infinity would be exhibited by the series or function which diverged the quickest. Consequently any function which diverged instantaneously, such as any number divided by zero, would most likely provide the largest infinity.

(iii) This is the most difficult of the above four questions to answer. In 1737 Leonhard Euler, in a corollary to a proof regarding infinite products, stated that the smallest of the infinities was $ln(\infty)$, being the series sum of $\zeta(1)$, [3]. However, in the same dissertation, he also showed that the series sum of the reciprocal primes was $lnln(\infty)$, which must be smaller than $ln(\infty)$. It is not known whether another divergent series exists that can be shown to diverge to a smaller infinity.

(iv) In view of the above, it is considered that the term infinity and the symbol ∞ should be reserved for the series S_2 in (5.1), which by default then represents the infinity of the term numbers in all other series. This would provide a definitive meaning to Euler's determinations in (iii) above. The degree of infinity for all other series that diverged would then need to be couched in terms of the attributes of each one.

Note that in (iii) and (iv) above, the Mascheroni Constant has been deliberately omitted.

6.0 Conclusions.

The results presented here with regard to Sections 2.0 and 4.0, are of course somewhat controversial in that they assign a finite series sum to what is generally regarded as divergent series. However, it is proposed that the proofs provided in those sections are sufficient, as well as necessary, to justify the results. In any case, the series sums so obtained result from an averaging technique and are therefore not in the strictest sense a proper series sum as would result from convergence.

The methods of estimation of series sums of the type of (3.1) is far quicker than that presented in [2]. However, it suffers from the deficiency wherein the degree of precision desired is not reached until the decimal place of interest permanently repeats. Hence the initial number of partial sums needed to attain the desired precision is not known beforehand. From that aspect, the method of [2] is superior.
The method here can be used with any series other than just those of type (3.1), providing term ratios meet the criteria stated in Section 4.3. But, its efficiency reduces with series containing an irregular series of terms. However, with such series it is the only method of obtaining a series sum other than just directly summing the series terms themselves.

On the nature of infinity, the results proposed here, to a limited degree, parallel the thinking of the eighteenth century mathematicians, and will therefore also be perceived as controversial. However, it is believed to be the only explanation that satisfies the results obtained in the determination of the series sum of the divergent alternating infinite series examined here. Accordingly, the various infinities to which the two halves of these series diverge must be, within the realm of all infinities, commutable, and can in certain circumstances, produce finite numbers. This also applies to certain convergent series such as $\eta(1)$ as shown in the main text. Also, within the realm of the infinities, it would be expected that they would also be associative and distributive, but proof of this has not been explored.

APPENDIX A.

Application to Other Series Expansions.

The process here to determine the series sums of the differentials of $ln(1 + x)$ can be extended to other functions that can be similarly expanded. As an example consider $tan^{-1}(x)$. Expanded this becomes

$$\boldsymbol{tan}^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \tag{A.1}$$

and when $x = 1$ it is well known that this series converges to $\pi/4$ as is confirmed by the left hand side.

Differentiating (A.1) gives

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots \tag{A.2}$$

and when $x = 1$ this becomes

$$\frac{1}{2} = 1 - 1 + 1 - 1 + 1 - \cdots \tag{A.3}$$

as per (2.3).

Further differentials produce identical results to those in Sections 2.0 and 4.1. Putting x to higher values than unity then produces results identical to Section 4.2.

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