THE CLOSED FORM OF CONVERGENT INFINITE SERIES.

6

SOME LOGARITHMIC INTEGRALS AND THE INFINITE SERIES THEY REPRESENT.

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ABSTRACT.

This paper presents three types of logarithmic integrals together with the infinite series they represent in closed form.

CONTENTS.

APPENDICES.

1.0 Introduction.

Four of the previous five papers in this series have dealt with convergent infinite series where closed forms were direct functions of the parameter π. In this paper the logarithmic function *ln(1 + x)*, and its Newtonian expansion, is explored for its capability to produce convergent infinite series. Three variations are investigated using the method described in [4]. Accordingly, this method, as it has in previous papers, allows each process to be generalised so that the closed form of any particular convergent infinite series in these families can be determined via a single application. It should be noted that the infinite series investigated here are only those with unity as the first term and with unity in the numerator of all other terms.

2.0 Some Logarithmic Integrals and the Infinite Series they Represent in Closed Form.

2.1 Integrals Involving the Group $\int_n \boldsymbol{ln}(1+x)dx$.

Note that in this Section the sign $\int_n (\;)\,dx$ is to be interpreted as the "n'tuple integration" of the enclosed function.

It is well known that Sir Isaac Newton's expansion of *ln*(1 + *x*) is given by

$$\boldsymbol{ln}(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \tag{2.1}$$

and putting $x = 1$ in (2.1) yields

$$\boldsymbol{ln}(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \tag{2.2}$$

which is thus the closed form of η(1). Now, integrating (2.1) yields

$$(1+x)\boldsymbol{ln}(1+x) - x = \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^5}{20} + \cdots \tag{2.3}$$

where the constant of integration is clearly zero, i.e. by putting $x = 0$. Multiplying (2.3) by 2 and then putting $x = 1$ gives the closed form of a new series

$$4\boldsymbol{ln}(2) - 2 = 1 - \frac{1}{3} + \frac{1}{6} - \frac{1}{10} + \cdots \tag{2.4}$$

Now, integrating (2.3) again gives

$$3(1+x)^2\,\boldsymbol{ln}(1+x) - \frac{9x^2}{2} - 3x = x^3 - \frac{x^4}{4} + \frac{x^5}{10} - \frac{x^6}{20} + \cdots \tag{2.5}$$

and with $x = 1$ this yields another new series

$$12\boldsymbol{ln}(2) - \frac{15}{2} = 1 - \frac{1}{4} + \frac{1}{10} - \frac{1}{20} + \cdots \tag{2.6}$$

Clearly this process can be continued ad infinitum to yield progressively the closed forms of all series in this group. The generalised result is presented in Section 3.

2.2 Integrals Involving the Group $\int \frac{\mathbf{ln}(1+x)}{x^m}dx$.

It was stated in [2] that the closed forms of ζ(*m*) and η(*m*-1) where *m* was an odd natural number was not currently attainable in analytical form because the integral

$$I = \int \frac{\mathbf{ln}(1+x)}{x}dx \tag{2.7}$$

did not appear to have a closed form solution itself. If (2.7) is integrated by parts, it self cancels, viz with

$$\frac{dv}{dx} = \mathbf{ln}(1+x) \qquad v = (1+x)\mathbf{ln}(1+x) - x$$

and (2.8)

$$u = \frac{1}{x} \qquad \frac{du}{dx} = -\frac{1}{x^2}$$

then

$$I = \frac{(1+x)\mathbf{ln}(1+x) - x}{x} + I + \int \frac{\mathbf{ln}(1+x)}{x^2}dx - \mathbf{ln}(x) \tag{2.9}$$

so that *I* is cancelled. However, (2.9) then reduces to

$$\int \frac{\mathbf{ln}(1+x)}{x^2}dx = 1 + \mathbf{ln}(x) - \frac{(1+x)\mathbf{ln}(1+x)}{x} \tag{2.10}$$

Consequently, if the right hand side of (2.1) is divided by x^2 and integrated, it produces in conjunction with (2.10)

$$1 + \mathbf{ln}(x) - \frac{(1+x)\mathbf{ln}(1+x)}{x} = \mathbf{ln}(x) - \frac{x}{2} + \frac{x^2}{6} - \frac{x^3}{12} + \frac{x^4}{20} - \cdots \tag{2.11}$$

and where by L'Hopitals Rule the constant of integration is zero.

Now after adding unity to both sides and putting $x = 1$, (2.11) reduces to the closed form of another convergent infinite series thus

$$2\{1 - \mathbf{ln}(2)\} = 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{12} + \frac{1}{20} - \cdots \tag{2.12}$$

This process ends here because a further integration of (2.11) would require evaluation of (2.7) within the integration of the left hand side. However, if the integration of (2.10) is attempted by parts, this time the integral only partly self cancels and it yields

$$\int \frac{\mathbf{ln}(1+x)}{x^3}dx = \frac{1}{2}\left\{\frac{(x^2-1)}{x^2}\mathbf{ln}(1+x) - \mathbf{ln}(x) - \frac{1}{x} - 1\right\} \tag{2.13}$$

so that dividing the right hand side of (2.1) by x^3 and integrating gives in conjunction with (2.13)

$$\frac{1}{2}\left\{\frac{(x^2-1)}{x^2}\mathbf{ln}(1+x) - \mathbf{ln}(x) - \frac{1}{x} - 1\right\} + k = -\frac{1}{x} - \frac{1}{2}\mathbf{ln}(x) + \frac{x}{3} - \frac{x^2}{8} + \frac{x^3}{15} - \cdots \tag{2.14}$$

Which reduces to

$$\frac{1}{2x}-\frac{1}{2}+\frac{\left(x^2-1\right)}{2x^2}\boldsymbol{ln}(1+x)+k=\frac{x}{3}-\frac{x^2}{8}+\frac{x^3}{15}-\cdots \qquad (2.15)$$

By L'Hopitals Rule the constant of integration is 1/4, so that inserting this into (2.15) and adding unity to both sides gives

$$\frac{3}{4}+\frac{1}{2x}+\frac{\left(x^2-1\right)}{2x^2}\boldsymbol{ln}(1+x)=1+\frac{x}{3}-\frac{x^2}{8}+\frac{x^3}{15}-\cdots \qquad (2.16)$$

and again putting $x = 1$ finally gives

$$\frac{5}{4}=1+\frac{1}{3}-\frac{1}{8}+\frac{1}{15}-\cdots \qquad (2.17)$$

Because of the nature of (2.16) it can be taken through one more integration process to produce the first Sub-Group of this Group. This is contained in Section 2.4. Here, the next entry in the main Group is derived via the integration of the left hand side of (2.13). This yields

$$\int\frac{\boldsymbol{ln}(1+x)}{x^4}dx=\frac{1}{3}\left\{1+\frac{1}{x}-\frac{1}{2x^2}+\boldsymbol{ln}(x)-\frac{\left(x^3+1\right)}{x^3}\boldsymbol{ln}(1+x)\right\}+k \qquad (2.18)$$

so that dividing the right hand side of (2.1) by x^4 and integrating, in conjunction with (2.18) after determination and insertion of the constant of integration, k, (= -2/9), and addition of unity to both sides gives

$$\frac{10}{9}-\frac{1}{6x}+\frac{1}{3x^2}-\frac{\left(x^3+1\right)}{3x^3}\boldsymbol{ln}(1+x)=1-\frac{x}{4}+\frac{x^2}{10}-\frac{x^3}{18}+\cdots \qquad (2.19)$$

Again putting $x = 1$ finally gives

$$\frac{23}{18}-\frac{2}{3}\boldsymbol{ln}(2)=1-\frac{1}{4}+\frac{1}{10}-\frac{1}{18}+\cdots \qquad (2.20)$$

It can be seen that the nature of the left hand side of (2.19) is such that it will allow two further integrations to produce the second Sub-Group. This is also contained in Section 2.4. The main process in this Group can clearly be continued ad infinitum to yield progressively the closed form of all convergent infinite series in the Group. The generalised result is presented in Section 2.3.

2.3 Integrals Involving the Group $\int x^m \boldsymbol{ln}(1+x)dx$.

Starting with the first member of this Group, it is easily shown via integration by parts that

$$\int x\,\boldsymbol{ln}(1+x)=\frac{1}{2}\left\{\left(x^2-1\right)\boldsymbol{ln}(1+x)-\frac{x^2}{2}+x\right\} \qquad (2.21)$$

where the constant of integration is zero. Therefore multiplying (2.1) by x and integrating yields in conjunction with (2.21) after adding unity to both sides

$$\frac{1}{2}\left\{\left(x^2-1\right)\boldsymbol{ln}(1+x)-\frac{x^2}{2}+x+2\right\}=1+\frac{x^3}{3}-\frac{x^4}{8}+\frac{x^5}{15}-\cdots \tag{2.22}$$

Now when x is put equal to unity, (2.22) yields the same result as (2.17). The right and left hand sides of (2.16) and (2.22) are however different so that this coincidental result only arises from this value of x, i.e. unity. Also note that (2.22) can be further integrated ad infinitum to form the first Sub-Group in this series.

Integration of the second member of the main Group yields

$$\int x^2\,\boldsymbol{ln}(1+x)=\frac{1}{3}\left\{\left(x^3+1\right)\boldsymbol{ln}(1+x)-\frac{x^3}{2}+\frac{x^2}{3}-x\right\} \tag{2.23}$$

where the constant of integration is zero. Therefore multiplying (2.1) by x^2 and integrating yields in conjunction with (2.23) after adding unity to both sides

$$\frac{1}{3}\left\{\left(x^3+1\right)\boldsymbol{ln}(1+x)-\frac{x^3}{2}+\frac{x^2}{3}-x+3\right\}=1+\frac{x^4}{4}-\frac{x^5}{10}+\frac{x^6}{18}-\frac{x^7}{28}+\cdots \tag{2.24}$$

which when x is put equal to unity becomes

$$\frac{2}{3}\boldsymbol{ln}(1+x)+\frac{13}{18}=1+\frac{1}{4}-\frac{1}{10}+\frac{1}{18}-\frac{1}{28}+\cdots \tag{2.25}$$

where the right hand side is the "mirror image" of (2.20) and from which it can of course be derived directly. This trend continues throughout the Group so that it is redundant apart from two facts thus,

(i) The results from each Group member can be further integrated to form an infinite Sub-group

and

(ii) When x .>1, an infinite number of different alternating term "divergent" series is produced.

2.4 Sub-Groups of Section 2.2.

The first Sub-group of Section 2.2 is obtained by integrating (2.16) to obtain

$$\frac{x}{4}-\frac{1}{2}+\frac{(x+1)^2}{2x}\boldsymbol{ln}(1+x)=x+\frac{x^2}{6}-\frac{x^3}{24}+\frac{x^4}{60}-\cdots \tag{2.26}$$

and again when $x = 1$ this reduces to

$$2\boldsymbol{ln}(2)-\frac{1}{4}=1+\frac{1}{6}-\frac{1}{24}+\frac{1}{60}-\cdots \tag{2.27}$$

This Sub-Group now ends with just this single entry because the evaluation of (2.7) would be required in the integration of the left hand side of (2.26).

The next Sub-Group of Section 2.2 is the integration of (2.19), i.e.

$$\frac{13x}{9}-\frac{1}{6x}+\frac{1}{12}=\frac{\left(2x^3+3x^2-1\right)}{6x^2}\boldsymbol{ln}(1+x)=x-\frac{x^2}{8}+\frac{x^3}{30}-\frac{x^4}{72}+\cdots \tag{2.28}$$

where the constant of integration, k =-1/12 has been inserted. Putting x = 1 in (2.28) gives

$$\frac{49}{36}-\frac{2}{3}\boldsymbol{ln}(2)=1-\frac{1}{8}+\frac{1}{30}-\frac{1}{72}+\cdots \tag{2.29}$$

Eq.(2.28) can be subjected to one further integration to yield

$$\frac{43}{18}-\frac{4}{3}\boldsymbol{ln}(2)=1+\frac{1}{2}-\frac{1}{24}+\frac{1}{120}-\frac{1}{360}+\cdots \tag{2.30}$$

This Sub-Group ends here for the same reason as above.

3.0 Group Generalisation Plus examples.

3.1 The Group $\int_n \boldsymbol{ln}(1+x)dx$.

From (2.1), (2.4) and (2.6) this Group can be generalised as follows
For the left hand side integration

$$I(m)=m\left[(1+x)^m \boldsymbol{ln}(1+x)-x(1+x)^{(m-2)}-\sum_{n=1}^{(m-2)}\frac{(m-1)!}{(m-n)!}\int_n x(1+x)^{(m-n-2)}dx\right] \tag{3.1}$$

where $\int_n(\ )dx$ is the n'tuple integration of the variable.
For the right hand side infinite series

$$S(m)=1+\sum_{n=1}^{\infty}\frac{(-1)^n m!\,n!\,x^{(m+n)}}{(m+n)!} \tag{3.2}$$

As an example of the application of (3.1) in conjunction with (3.2), consider the infinite series closed form result when m = 6, then from (3.1)

$$I(6)=6\left[(1+x)^6 \boldsymbol{ln}(1+x)-x(1+x)^4-\int_1 x(1+x)^3 dx-5\int_2 x(1+x)^2 dx-20\int_3 x(1+x)dx-60\int_4 xdx\right] \tag{3.3}$$

and with x = 1 this evaluates to

$$I(6)=192\boldsymbol{ln}(2)-\frac{661}{5} \tag{3.4}$$

For the right hand side from (3.2)

$$S(6)=1-\frac{6!\,x^7}{7!}+2\frac{6!\,x^8}{8!}-6\frac{6!\,x^9}{9!}+\cdots \tag{3.5}$$

and with x = 1 this becomes

$$S(6)=1-\frac{1}{7}+\frac{1}{28}-\frac{1}{84}+\cdots \tag{3.6}$$

The final result is then the equality of (3.4) and (3.6), viz

$$192\,\boldsymbol{ln}(2)-\frac{661}{5}=1-\frac{1}{7}+\frac{1}{28}-\frac{1}{84}+\cdots \tag{3.7}$$

3.2 The Group $\int_n \frac{\boldsymbol{ln}(1+x)}{x^m}dx$.

From(2.12), (2.16) and (2.19) this Group can be generalised as follows. For the left hand side integral

$$I(m)=1+\frac{(-1)^m}{(m-1)^2}+\sum_{n=1}^{(m-2)}\frac{(-1)^{(n-1)}}{n(m-1)\,x^{(m-n-1)}}+(-1)^{(m-1)}\left\{\frac{x^{(m-1)}+(-1)^m}{(m-1)\,x^{(m-1)}}\right\}\boldsymbol{ln}(1+x) \tag{3.8}$$

where $m > 1$.

For the right hand side infinite series

$$S(m)=1+\sum_{n=1}^{\infty}\frac{(-1)^{(m+n)}x^n}{n(n+m-1)} \tag{3.9}$$

where m is the power of x in the denominator of the generating integral, i.e. as in (2.10) and (2.13) etc. As an example of the application of (3.8) in conjunction with (3.9), consider the infinite series closed form result when $m = 5$.

$$I(5)=1-\frac{1}{16}+\frac{1}{4x^3}-\frac{1}{8x^2}+\frac{1}{12x}+\frac{(x^4-1)}{4x^4}\boldsymbol{ln}(1+x) \tag{3.10}$$

and when $x = 1$, this evaluates to

$$I(5)=\frac{55}{48} \tag{3.11}$$

For the right hand side

$$S(5)=1+\frac{x}{5}-\frac{x^2}{12}+\frac{x^3}{21}-\cdots \tag{3.12}$$

and with $x = 1$, this becomes

$$S(5)=1+\frac{1}{5}-\frac{1}{12}+\frac{1}{21}-\cdots \tag{3.13}$$

The final result is then the equality of (3.11) and (3.13)

$$\frac{55}{48}=1+\frac{1}{5}-\frac{1}{12}+\frac{1}{21}-\cdots \tag{3.14}$$

4.0 Conclusions.

From the development presented here it is clear that the infinite series contained in these Groups form a series of families, the controlling attribute of which is the generating integral. Mostly their closed forms are all functions of *ln*(2), but some have been shown to be solely rational numbers. However, this is only so when the value of the independent variable, *x* is unity. When *x* is greater than unity and natural, all series have closed forms that are logarithmic and therefore irrational. For a closed form to be rational when $x \neq 1$, x must be non-natural. It should also be noted that this condition also means that it is possible to derive many infinite series for which the closed form is zero. Such determination however, cannot be effected analytically, but must be pursued numerically.

From the nature of the generating integrals, it is clear that the closed forms of a virtually unlimited number of convergent series can be derived, a statement which applies to each Group and in limited form to each Sub-Group. However, it is noted that all such series are composed of terms that alternate in sign, {a consequence of the expansion of the base function, $ln(1+x)$}. To obtain series that contain only terms of the same sign would require a different base function. In addition to the series resulting from the Zeta function, some of the series derived in [1] are an example.

The extent of convergent infinite series is so vast that those explored in the six papers presented here, [1] → [5] plus this one, represent only a fraction, and the subject will require extensive study before a fuller understanding of all their attributes can be said to have been achieved.

APPENDIX A.

Summary of Group and Sub-Group Results.

A.1 The Group Generated by the Integral $\int_n \boldsymbol{ln}(1+x)dx$.

m	Integral	Closed Form, (x = 1)	Infinite Series
1	$ln(1+x)$	$ln(2)$	$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\cdots$
2	$\int_1 \boldsymbol{ln}(1+x)dx$	$4ln(2) - 2$	$1-\frac{1}{3}+\frac{1}{6}-\frac{1}{10}\cdots$
3	$\int_2 \boldsymbol{ln}(1+x)dx$	$12ln(2) -15/2$	$1-\frac{1}{4}+\frac{1}{10}-\frac{1}{20}\cdots$
4	$\int_3 \boldsymbol{ln}(1+x)dx$	$32ln(2) - 64/3$	$1-\frac{1}{5}+\frac{1}{15}-\frac{1}{35}\cdots$
5	$\int_4 \boldsymbol{ln}(1+x)dx$	$80ln(2) - 655/12$	$1-\frac{1}{6}+\frac{1}{21}-\frac{1}{56}\cdots$
6	$\int_5 \boldsymbol{ln}(1+x)dx$	$192ln(2) - 661/5$	$1-\frac{1}{7}+\frac{1}{28}-\frac{1}{84}\cdots$
etc.			

Table A.1 - Summary of Infinite Series Closed Forms Produced by the Integral $\int_n \boldsymbol{ln}(1+x)dx$.

Note that in addition to the generalised algorithm for the terms of the infinite series, (3.2), the denominators are related in that the denominator for any particular term is the sum of the denominator of the previous term in the same row and the denominator in the same column in the row above, i.e. row 6 column 4 = 84 =28 + 56.
Also, note that the result for m = 2 can be manipulated to

$$\frac{4}{3}\boldsymbol{ln}(2)=1-\frac{1}{9}+\frac{1}{18}-\frac{1}{30}+\cdots \qquad \text{(A.1)}$$

and this result can be further manipulated to give

$$2\,\boldsymbol{ln}(2) = 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{12} - \frac{1}{20} + \cdots \tag{A.2}$$

Obviously all other series in this Group can be similarly manipulated to give a new series involving *ln*(2). However, these do not conform to the criteria stated at the end of the Introduction.

A.2 The Group Generated by the Integral $\int \frac{\boldsymbol{ln}(1+x)}{x^m}dx$.

m	Integral	Closed Form, (*x* = 1)	Infinite Series
2	$\int \boldsymbol{ln}(1+x)\big/ x^2\, dx$	2{1 - *ln*(2)}	$1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{12} + \frac{1}{20} - \cdots$
3	$\int \boldsymbol{ln}(1+x)\big/ x^3\, dx$	5/4	$1 + \frac{1}{3} - \frac{1}{8} + \frac{1}{15} - \frac{1}{24} + \cdots$
4	$\int \boldsymbol{ln}(1+x)\big/ x^4\, dx$	23/18 - (2/3)*ln*(2)	$1 - \frac{1}{4} + \frac{1}{10} - \frac{1}{18} + \frac{1}{28} - \cdots$
5	$\int \boldsymbol{ln}(1+x)\big/ x^5\, dx$	55/48	$1 + \frac{1}{5} - \frac{1}{12} + \frac{1}{21} - \frac{1}{32} + \cdots$
6	$\int \boldsymbol{ln}(1+x)\big/ x^6\, dx$	347/300 - (2/5)*ln*(2)	$1 - \frac{1}{6} + \frac{1}{14} - \frac{1}{24} + \frac{1}{36} - \cdots$
7	$\int \boldsymbol{ln}(1+x)\big/ x^7\, dx$	397/360	$1 + \frac{1}{7} - \frac{1}{16} + \frac{1}{27} - \frac{1}{40} + \cdots$
etc.			

Table A.2 - Summary of Infinite Series Closed Forms
Produced by the Integral $\int \boldsymbol{ln}(1+x)/x^m dx$.

It is clear from the above table that the denominators in the terms of the infinite series are also pattern related. Also in this table note that the series for odd *m* can all be converted to one with a closed from of zero. They do not however, conform to the criteria stated at the end of the Introduction.
Also note that the series for *m* = 2 can be converted to

$$\boldsymbol{ln}(2) = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{12} + \frac{1}{24} - \frac{1}{40} + \cdots \tag{A.3}$$

which converges more quickly than (2.2). It also shows that there can be many different series with identical closed forms.

A.3 The Sub-Groups Generated by the Integral $\int \frac{\boldsymbol{ln}(1+x)}{x^m}dx$.

m	Integral	Sub-Group	Closed Form, (x = 1)	Infinite Series
3	$\int \boldsymbol{ln}(1+x) \big/ x^3 \, dx$	1	2*ln*(2) - 1/4	$1+\frac{1}{6}-\frac{1}{24}+\frac{1}{60}\cdots$
4	$\int \boldsymbol{ln}(1+x) \big/ x^4 \, dx$	1	49/36 - (2/3)*ln*(2)	$1-\frac{1}{8}+\frac{1}{30}-\frac{1}{72}\cdots$
		2	43/18 - (4/3)*ln*(2)	$1+\frac{1}{2}-\frac{1}{24}+\frac{1}{120}\cdots$
etc.				

Table A.3 - Summary of Infinite Series Closed Forms
Produced by Sub-Groups of the Integral $\int \boldsymbol{ln}(1+x)/\,x^m dx$.

It is clear from the text and the above table that as *m* increases by unity, the number of Sub-Groups also increase by unity. A clear relationship between the denominators of the terms in the Sub-Group series is not obvious here and probably needs derivation of more Sub-Group series before one becomes apparent.

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