THE CLOSED FORMS OF CONVERGENT INFINITE SERIES

5

ESTIMATION OF THE SERIES SUM OF NON-CLOSED FORM ALTERNATING SERIES TO A HIGH DEGREE OF PRECISION.

Peter G.Bass.

M14 Ver. 1.0.0.
www.relativitydomains.com
May 2014

Abstract

This paper provides a numerical method for estimating the value of the series sum of any alternating infinite series, for any value of the exponent *m*, for which a closed form does not exist. In particular, it covers the series sums of the opposing exponent *m* Zeta and Eta series, (odd exponents), and the Xi series (even exponents).

Contents.

1.0 Introduction.

In 1730 Leonhard Euler developed his first of three solutions to the Basel problem, the closed form of ζ(2). His first was the most productive as it also provided closed form solutions for ζ(*m*) and η(*m*) for all even *m*, and ξ(*m*) for all odd *m*. However, none of his methods provided closed form solutions for any of the above series for the opposing values of *m*. In the intervening 280 plus years, despite the attention of many proficient mathematicians, this problem has remained unsolved. A brief resume of the difficulty associated with it is given in the next Section.

Consequently, it is considered that the next best alternative, is to provide a means to estimate these closed form values to as high a degree of precision as possible with the minimum amount of computation.

2.0 The Difficulties Associated with the Solution of the Closed Forms of ζ(*m*), η(*m*) and ξ(*m*) for Opposing *m*.

The closed forms of ζ(*m*) and η(*m*) for even *m* and ξ(*m*) for odd *m*, are all direct function of π^m and are therefore amenable to derivation via analysis of suitable circular functions. This was Euler's approach in his three methods, and in the extension of his first two methods in [1] and [2], plus the methods described in [3] and [4]. However, these series with opposing values of *m* are not direct functions of π, and are therefore not amenable to derivation via analysis of circular functions. As an example, consider η(3). If η(3) were to be evaluated via the method of [4], then the following relationship would need to exist

$$F(\theta) = \cos\theta - \frac{\cos 2\theta}{2^3} + \frac{\cos 3\theta}{3^3} - \frac{\cos 4\theta}{4^3} + \cdots \qquad (2.1)$$

Differentiating (2.1) twice to obtain the Base equation, (as described in [4]), would give

$$\frac{d^2F(\theta)}{d\theta^2} = f(\theta) = \cos\theta - \frac{\cos 2\theta}{2} + \frac{\cos 3\theta}{3} - \frac{\cos 4\theta}{4} + \cdots \qquad (2.2)$$

But, putting θ = 0 in (2.2) gives η(1) on the right hand side, the closed form of which is well known to be ln(2). While it is possible to construct a logarithmic function in place of *f*(θ) that satisfies (2.2), it cannot be derived via a Fourier series expansion, and cannot therefore be integrated up to give η(3). It would only satisfy η(1).

To provide a relationship that could theoretically be integrated, recourse could be made to Newton's expansion of ln(1 + *x*), viz

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad (2.3)$$

which, as is well known, by putting $x = 1$, gives ln(2) for the closed form of η(1). Now, to integrate (2.3) up to give η(3), would initially mean evaluating the integral

$$\int \frac{\ln(1+x)}{x}dx = x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \cdots \qquad (2.4)$$

to give the closed form of η(2) by again putting x = 1. However, the left hand side of (2.4) is a polylogarithmic integral which doe not appear to have a closed form solution. Integration by conventional substitution methods, (i.e. $1 + x = e^y$) only permits a solution as an infinite series which converges more slowly than η(2). If (2.4) is integrated by parts, in the result the integral itself cancels leaving other terms which do not provide the required solution. Other means of solving this integral are not apparent, and hence the derivation of the closed forms of the opposing *m* Zeta, Eta and Xi series remains at present unattainable.

3.0 Derivation of a Numerical Algorithm to Estimate the Closed Form Values of the Opposing *m* Zeta, Eta and Xi Series et al.

3.1 The Algorithm Derivation.

For any infinite series that does not have a closed form solution, to estimate its summed value to any desired precision, it is necessary to sum a sufficient number of terms until the applicable decimal place stops changing. For very slowly converging series this can mean summing many thousands of terms. With the computing powers that are available today, this would not be too difficult a task. However, it is still useful to reduce the computing requirements where possible. To that end, the following process will, in extreme cases reduce these requirements by up to 65 times.

If the partial sums of an alternating infinite series are plotted in general terms, they can be represented as in Fig. 3.1 below

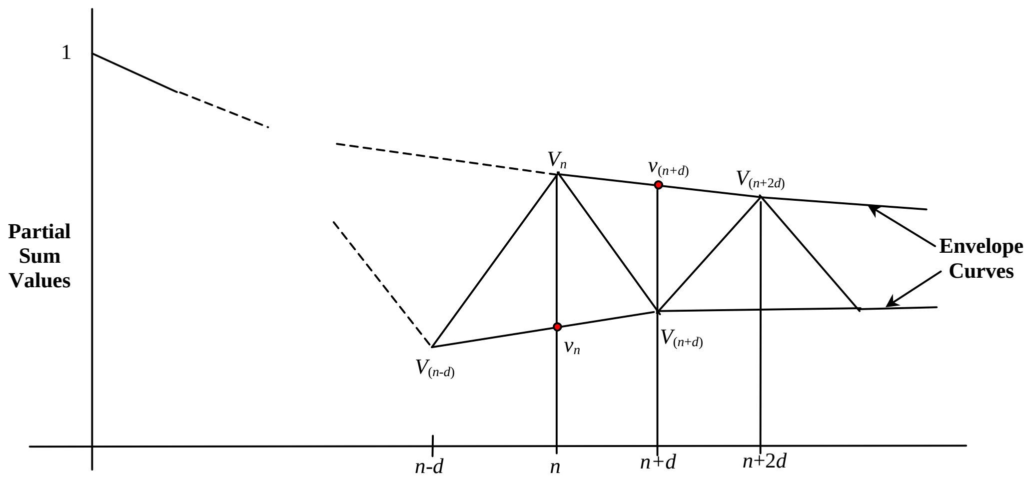

Fig. 3.1 - Partial Sums vs Term Numbers for an Alternating Infinite Series.

In Fig. 3.1,

$V_{(n \pm d)}$	is the $(n \pm d)$th partial sum.
$v_n, v_{(n+d)}$	are inter-partial sum values on the envelope curves.
d	is the series term denominator separation constant.

Also note that

$$V_{(n-d)} = V_n - \frac{1}{n^m}$$

$$V_{(n+d)} = V_n - \frac{1}{(n+d)^m} \tag{3.1}$$

$$V_{(n+2d)} = V_n - \frac{1}{(n+d)^m} + \frac{1}{(n+2d)^m} = V_n - \frac{(n+2d)^m - (n+d)^m}{(n+2d)^m (n+d)^m}$$

Where *m* is the series exponent.

To begin, determine an estimate of the series sum based upon the odd term partial sum V_n.
First

$$v_n = \frac{V_{(n+d)} - V_{(n-d)}}{2} + V_{(n-d)} \tag{3.2}$$

So that the estimate is

$$O_n = \frac{V_n + v_n}{2} = \frac{V_n + V_{(n-d)} + \frac{V_{(n+d)} - V_{(n-d)}}{2}}{2} \tag{3.3}$$

Which reduces to, via substitution of terms from (3.1)

$$O_n = V_n - \frac{1}{4}\left\{\frac{(n+d)^m + n^m}{n^m (n+d)^m}\right\} \tag{3.4}$$

Now determine an estimate of the series sum based upon the even term partial sum $V_{(n+d)}$.
First

$$v_{(n+d)} = V_n - \frac{\left(V_n - V_{(n+2d)}\right)}{2} \tag{3.5}$$

So that the estimate is

$$E_{(n+d)} = \frac{V_{(n+d)} + v_{(n+d)}}{2} = \frac{V_{(n+d)} + V_n - \frac{V_n - V_{(n+2d)}}{2}}{2} \tag{3.6}$$

Which reduces to, via substitution of terms from (3.1)

$$E_{(n+d)} = V_n - \frac{1}{4}\left\{\frac{3(n+2d)^m - (n+d)^m}{(n+2d)^m (n+d)^m}\right\} \tag{3.7}$$

$E_{(n+d)}$ will always be greater than O_n, and for low to medium values of *n*, $E_{(n+d)}$ will be greater than the closed form value, and O_n, lower. Therefore, subtracting (3.4) from (3.7) yields

$$\varepsilon = \frac{1}{4}\left\{\frac{(n+d)^m + n^m}{(n+d)^m n^m} - \frac{3(n+2d)^m - (n+d)^m}{(n+2d)^m (n+d)^m}\right\} \tag{3.8}$$

and this reduces to

$$\varepsilon = \frac{1}{4}\left\{ \frac{1}{n^m} - \frac{2}{(n+d)^m} + \frac{1}{(n+2d)^m} \right\} \tag{3.9}$$

ε is a measure of the range width within which the series sum lies. Now (3.9) can easily be solved in terms of n when m is very small, i.e. single digit. Even so, it is still easier to simply plug values of n into (3.9) to find the value of ε desired. Then that number for n is used to determine O_n and $E_{(n+d)}$ so giving the partial sum range within which the series sum lies. The series sum estimate, to the desired degree of precision, is then given by the average of O_n and $E_{(n+d)}$. The brief table below gives some random figures of minimum expected precision for $\eta(m)$ using this method.

m	***n*, (*d* = 1)**	**ε**
3	49	9.6E-9
7	29	7.2E-13
11	18	4.2E-16
21	7	4.0E-19

Table 3.1 - Estimated Precision for Various *m* and *n* for η(*m*).

As a specific example, to determine η(3) to a precision of <1E-8, using conventional methods requires summing 465 terms. As seen from the Table above, the method here requires only 49 terms to achieve the same or better degree of precision, a reduction of some 9.5 times. In fact, using convention summing methods to achieve exactly the same precision as obtained here, would require summing 1588 terms.
It should also be noted that this method can be used with any alternating convergent infinite series for which a closed form is not available and for which the series term denominator separation is a constant. Appendix A provides some examples for determining the series sum of η(*m*), (and therefore ζ(*m*)), for odd *m*, and for ξ(*m*) for even *m*, plus other series.

3.2 Limitations.

As stated earlier, for low to medium values of n, O_n and $E_{(n+d)}$ bracket the closed form value of the series under investigation, and so the precision of its estimate is at least equal to the value of ε selected, and in most cases much better. However, when n is increased to high values, the closed form value can fall just outside of the range O_n to $E_{(n+d)}$. When this happens the precision of the estimate given by the average of O_n and $E_{(n+d)}$ will be less than the value of ε. The precision will then remain constant, irrespective of any increased value of n and associated ε, at the level achieved when O_n and $E_{(n+d)}$ last bracketed the closed form value.

Using the example of η(3), the precision obtained for n = 49 is 2.3999...E-10, against a selected ε of 1E-8. When n is increased to 887 to give an ε of 6.841...E-15 the closed form drops just outside the associated range of O_n and $E_{(n+d)}$, and a slightly lower precision of 6.994...E-15 is obtained. As n is increased further this level of precision remains substantially constant irrespective of the level of ε specified by the increasing value of n.
The figures in this example only apply to the value of the exponent m = 3. When m is higher than this value, the limiting level of precision is well in excess of 1E-15.

4.0 Conclusions.

While this method provides a simple means of estimating the series sum of applicable alternating infinite series, it still requires summing, in some cases, a considerable number of terms of the series. However, in extreme cases it can mean summing over 60 times less than the number of terms summed conventionally to achieve the same degree of precision. Also, it provides very quickly, prior to any large scale computation, the number of terms needed to obtain the desired precision, so avoiding monitoring a particular decimal place for its constancy in a conventional summing process as more terms are added to the sum.

The values that can be calculated using this method, being limited to a certain number of decimal places is in fact no different to those series for which a closed form is available. This is so because in most cases, such closed forms are either functions of π or logarithmic, both of which being irrational can themselves only be expressed to a limited number of decimal places. That is not to say that the solutions obtained here are equivalent to those determined from a closed form. The solutions obtained here are not those of a closed form, but of a series sum.

The main limitation of the method, apart from the necessity to still sum a number of terms of the series, is that of the bottoming out of the level of precision that can be obtained as the value of n is increased. However, this limitation is considered only a minor one as the level of n that causes this effect is very high, and therefore even for the minimum value of m, the minimum degree of precision so obtained is still extremely high.

The lack of solutions of these series in closed form does not mean that such solutions do not exist in the form of elementary functions. It is believed that such solutions do exist and as discussed in the text above, are expected to be of logarithmic or polylogarithmic form. Their evaluation, in conjunction with the method of [4], appears to depend upon the derivation of a closed form solution to the integral depicted by Eq.(2.4), and this determination may require the development of some "new maths".

Finally, although redundant, this method is of course also applicable to those series for which a closed form is available.

Appendix A.

Application Examples for η(m), ζ(*m*), (*m* odd), and ξ(*m*), (*m* Even), Plus Other Applicable Series.

(i) Series sum estimate of $\eta(3)$ and $\zeta(3)$ to a minimum precision of 1E-8.

In this case $d = 1$. Therefore (3.9) becomes

$$\varepsilon = \frac{1}{4}\left\{\frac{1}{n^3} - \frac{2}{(n+1)^3} + \frac{1}{(n+2)^3}\right\} \tag{A.1}$$

It is important to note that the values of n chosen to insert into (A.1) to obtain the required ε must be values that provide a positive term in the series. Thus substitution of trial values of n into (A.1) to obtain a value of $\varepsilon \le$ 1E-8 yield $n = 49$, and so

$$V_{49} = 0.90154679728983 \qquad (A.2)$$

Then O_{49} becomes

$$O_{49} = V_{49} - \frac{1}{4}\left(\frac{50^3 + 49^3}{50^3 x 49^3}\right) \qquad (A.3)$$

Which, upon insertion of (A.2) becomes

$$O_{49} = 0.901542672324892 \qquad (A.4)$$

Similarly E_{50} evaluates to

$$E_{50} = 0.90154268193499 \qquad (A.5)$$

and the average of (A.4) and (A.5) is

$$\eta(3) \approx 0.901542677129695 \qquad (A.6)$$

and this is within -2.39992692385727E-10 of η(3) calculated to 15 places.

ζ(3) is then calculated as

$$\varsigma(3) \approx \frac{2^2}{2^2 - 1}\eta(3) = \frac{4}{3}\eta(3) = 1.2025690283959 \qquad (A.7)$$

which is within -3.22376347838826E-10 of ζ(3) calculated to 15 places.

<u>(ii) Series sum estimate of ξ(4) to a precision of 1E-8.</u>

In this case $d = 2$ and for an ε of 1E-8, the value of n required from (3.9) is 37 to give

$$V_{37} = 0.988944789892734 \qquad (A.8)$$

$$O_{37} = 0.988944548435577 \qquad (A.9)$$

$$E_{38} = 0.988944554172072 \qquad (A.10)$$

To give

$$\xi(4) \approx 0.988944551303824 \qquad (A.11)$$

Which is within -4.37280434084641E-10 of ξ(4) calculated to 15 places.

<u>(iii) Series sum estimate, to a precision of 1E-8, of.</u>

$$S(5) = 1 - \frac{1}{5^5} + \frac{1}{9^5} - \frac{1}{13^5} + \frac{1}{17^5} - \cdots \qquad (A.12)$$

In this case d = 4 and the required n is 25 to give

$$V_{25} = 0.999694803641416 \tag{A.13}$$

$$O_{25} = 0.999694765852922 \tag{A.14}$$

$$E_{26} = 0.999694773464016 \tag{A.15}$$

To give

$$S(5) \approx 0.999694769658469 \tag{A.16}$$

Which is within -1.69584235543141E-9 of S(5) calculated to 15 places.

(iv) Series sum estimate, to a precision of 1E-12, of.

$$S(6) = 1 - \frac{1}{6^6} + \frac{1}{11^6} - \frac{1}{16^6} + \frac{1}{21^6} - \cdots \tag{A.17}$$

In this case d = 5 and the required n is 61 to give

$$V_{61} = 0.9999790806374160 \tag{A.18}$$

$$O_{61} = 0.9999790806295390 \tag{A.19}$$

$$E_{61} = 0.9999790806302940 \tag{A.20}$$

To give

$$S(6) \approx 0.9999790806299160 \tag{A.21}$$

Which is within -1.10689235555128000E-13 of S(6) calculated to 15 places.

References.

[1] P.G.Bass, *The Closed Form of Convergent Infinite Series - 1 - A Extension of Leonhard Euler's First Method*, www.relativitydomains.com.

[2] P.G.Bass, *The Closed Form of Convergent Infinite Series - 2 - A Extension of Leonhard Euler's Second Method*, www.relativitydomains.com.

[3] P.G.Bass, *The Closed Form of Convergent Infinite Series - 3 - Generalisation Via Fourier Series Expansion*, www.relativitydomains.com.

[4] P.G.Bass, *The Closed Form of Convergent Infinite Series - 4 - Determination Via Recursive Integration*, www.relativitydomains.com.