THE CLOSED FORM OF CONVERGENT INFINITE SERIES

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DETERMINATION VIA RECURSIVE INTEGRATION.

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M13 Version 1.0.0.
March 2014

Abstract.

This paper demonstrates the determination of the closed form of the Zeta and Xi, (and Eta), infinite series via a process of recursive integration.

Contents.

1.0 Introduction.

It is well known that if an infinite series is convergent, it may be either differentiated or integrated term by term. Consequently, this attribute may be used to determine the closed form of convergent infinite series by repeated integration. In particular, the closed forms of the Zeta and Xi series, for all values of the exponent *m* may be determined. For the Zeta series only even values of *m* are applicable, and for the Xi series, only odd values. There are only two requirements. They are (i) the Base equation must conform to a Fourier series expansion, and (ii) the closed forms of the Eta function, for even values of *m*, must be known beforehand. This latter requirement allows the process to be generalised and thereby any closed form, for any applicable *m*, of the Zeta and Xi series to be immediately determined from either of just two equations. Without this prior knowledge, determination of the appropriate Eta closed forms can however, be determined during the integration process, and development of the closed forms of the other series is then effected in the form of a ladder function.

2.0 The Recursive Integration Process and Generalisation.

For the series concerned here, the Base equation is given by the Fourier series expansion of the trigonometrical parameter θ. Thus

$$\theta = 2\left(\sin\theta - \frac{\sin 2\theta}{2} + \frac{\sin 3\theta}{3} - \frac{\sin 4\theta}{4} + \cdots\right) \tag{2.1}$$

Integrate with respect to θ

$$\frac{\theta^2}{2} = -2\left(\cos\theta - \frac{\cos 2\theta}{2^2} + \frac{\cos 3\theta}{3^2} - \frac{\cos 4\theta}{4^2} + \cdots\right) + k \tag{2.2}$$

Putting θ = 0 yields the constant of integration, *k*, as

$$k = 2\left(1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\right) = 2\eta(2) \tag{2.3}$$

and thus

$$\frac{\theta^2}{2} = 2\eta(2) - 2\left(\cos\theta - \frac{\cos 2\theta}{2^2} + \frac{\cos 3\theta}{3^2} - \frac{\cos 4\theta}{4^2} + \cdots\right) \tag{2.4}$$

Now integrate again

$$\frac{\theta^3}{6} = 2\theta\eta(2) - 2\left(\sin\theta - \frac{\sin 2\theta}{2^3} + \frac{\sin 3\theta}{3^3} - \frac{\sin 4\theta}{4^3} + \cdots\right) + k \tag{2.5}$$

Putting θ = 0 yields the constant of integration, *k*, as zero, and so

$$\frac{\theta^3}{6} = 2\theta\eta(2) - 2\left(\sin\theta - \frac{\sin 2\theta}{2^3} + \frac{\sin 3\theta}{3^3} - \frac{\sin 4\theta}{4^3} + \cdots\right) \tag{2.6}$$

Integrate once more

$$\frac{\theta^4}{24} = \theta^2\eta(2) + 2\left(\cos\theta - \frac{\cos 2\theta}{2^4} + \frac{\cos 3\theta}{3^4} - \frac{\cos 4\theta}{4^4} + \cdots\right) + k \tag{2.7}$$

Putting θ = 0 yields the constant of integration, *k*, as

$$k = -2\left(1 - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \cdots\right) = -2\eta(4) \tag{2.8}$$

So that

$$\frac{\theta^4}{24} = \theta^2\eta(2) - 2\eta(4) + 2\left(\cos\theta - \frac{\cos 2\theta}{2^4} + \frac{\cos 3\theta}{3^4} - \frac{\cos 4\theta}{4^4} + \cdots\right) \tag{2.9}$$

Continuing in this fashion ad infinitum, enables two generalised equations to be established, one for *m* even, and one for *m* odd. They are

(i) For *m* even

$$\frac{\theta^m}{2m!} = \sum_{x=1}^{(m-1)} \frac{(-1)^{(m+x-1)}\eta(2x)\,\theta^{(m-2x)}}{(m-2x)!} - \sum_{y=1}^{\infty} \frac{(-1)^{\left(\frac{m}{2}-y\right)}\cos(y\theta)}{y^m} \tag{2.10}$$

(ii) For *m* odd

$$\frac{\theta^m}{2m!} = \sum_{x=1}^{(m-1)} \frac{(-1)^{(m+x)}\eta(2x)\,\theta^{(m-2x)}}{(m-2x)!} + \sum_{y=1}^{\infty} \frac{(-1)^{\left(\frac{m+1}{2}-y\right)}\sin(y\theta)}{y^m} \tag{2.11}$$

From these two equations the closed forms of ζ(*m*) for *m* even, (Eq.(2.10)), and ξ(*m*) for *m* odd, (Eq.(2.11)) can be determined.

3.0 Examples of Application.

3.1 For Zeta(*m*).

Example for *m* = 6, from (2.10)

$$\frac{\theta^6}{2x6!} = \frac{\eta(2)\theta^4}{4!} - \frac{\eta(4)\theta^2}{2!} + \eta(6) - \left(\cos\theta - \frac{\cos 2\theta}{2^6} + \frac{\cos 3\theta}{3^6} - \cdots\right) \tag{3.1}$$

Now put θ = π to give

$$\frac{\pi^6}{2x6!} = \left(\frac{\pi^2}{12}\right)\frac{\pi^4}{4!} - \left(\frac{7\pi^4}{720}\right)\frac{\pi^2}{2!} + \frac{31\pi^6}{30{,}240} - \left(-1 - \frac{1}{2^6} - \frac{1}{3^6} - \cdots\right) \tag{3.2}$$

and thus

$$\varsigma(6) = \frac{\pi^6}{30{,}240}(21 - 105 + 147 - 31) = \frac{\pi^6}{945} \tag{3.3}$$

3.2 For Xi(*m*).

Example for m = *7*, from (2.11)

$$\frac{\theta^7}{2x7!}=\frac{\eta(2)\theta^5}{5!}-\frac{\eta(4)\theta^3}{3!}+\frac{\eta(6)\theta}{1!}+\left(-\boldsymbol{sin}\,\theta+\frac{\boldsymbol{sin}\,2\theta}{2^7}-\frac{\boldsymbol{sin}\,3\theta}{3^7}-\cdots\right) \tag{3.4}$$

Now put θ = π/2 to give

$$\frac{\pi^7}{2x7!}=\frac{\left(\frac{\pi^2}{12}x\frac{\pi^5}{32}\right)}{5!}-\frac{\left(\frac{7\pi^4}{720}x\frac{\pi^3}{8}\right)}{3!}+\frac{31\pi^6}{30{,}240}x\frac{\pi}{2}-\left(1-\frac{1}{3^7}+\frac{1}{5^7}\cdots\right) \tag{3.5}$$

and thus

$$\xi(7)=\frac{\pi^7}{1{,}290{,}240}\left(-1+28-261\frac{1}{3}+661\frac{1}{3}\right)=\frac{61\pi^7}{184{,}320} \tag{3.6}$$

3.3 For η(*m*).

From the above it is clear that to determine the closed forms for Zeta(*m*) and Xi(*m*), it is necessary to know the closed forms for Eta(*m*) for all smaller even values of *m*. However, if this is not the case Eta(*m*) can be determined during the integration process as follows

In (2.4), now put θ = π/2 to give

$$\frac{\pi^2}{8}=2\eta(2)-2\left(\frac{1}{2^2}-\frac{1}{4^2}+\frac{1}{6^2}\cdots\right) \tag{3.7}$$

and therefore

$$\frac{\pi^2}{8}=2\eta(2)-\frac{1}{2}\eta(2) \tag{3.8}$$

so that

$$\eta(2)=\frac{\pi^2}{8}x\frac{2}{3}=\frac{\pi^2}{12} \tag{3.9}$$

and this simple exercise at (2.4), (2.9) etc produces the closed form for η(*m*) for all even *m*.

4.0 Conclusions

In comparing the process described here with that in [1], it is clear that the superiority of one over the other depends upon prior knowledge. If η(*m*) for all even *m* is known, then the method here is the superior because only one equation needs to be evaluated to determine Zeta(*m*) or Xi(*m*) for any applicable value of *m*. On the other hand, if η(*m*) is not known then either method is suitable as they both require calculation via a ladder process to obtain the desired result. In this latter case, the degree of calculation required in each method is similar but far less than any other known method.

Finally, this method is of course fully applicable to any convergent infinite series provided, (i) the Base equation conforms to a Fourier series expansion and (ii) the constants of integration can be evaluated.

References.

[1] P.G.Bass, *The Closed Forms of Convergent Infinite Series - 3 - Generalisation Via Fourier Series Expansion*, www.relativitydomains.com.