THE CLOSED FORMS OF CONVERGENT INFINITE SERIES

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GENERALISATION VIA FOURIER SERIES EXPANSION.

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Abstract.

The closed forms of the Zeta, Eta and Xi functions are herein fully generalised using Fourier Series Expansion.

Contents.

1.0 Introduction.

The extension of Leonhard Euler's first and second methods of solving the Basel Problem, as depicted in [1] and [2], are largely of academic interest only because of the severe limitations associated with each, primarily the computational burden with increasing values of the exponent.
In this paper such limitations are eliminated and a practical method for the determination of the closed forms of the Zeta and Eta functions with even exponents and for the Xi function with odd exponents is developed. This is effected via a generalised Fourier series expansion of a parameter that is known to produce such results. Substantial examples of application are also presented.

2.0 Generalisation of the Closed Forms of Convergent Infinite Series.

2.1 The Zeta Function, (*m* Even).

It is well known that if the trigonometrical parameter θ is subjected to a Fourier series expansion, the result leads directly to the closed form for Xi(1). Similarly, a Fourier series expansion of θ^2 leads to the closed form for ζ(2). To generalise these results, a Fourier series expansion of initially θ^m, with m an even natural number, is effected below.

With m even, θ^m is an even function and so a Fourier series expansion of θ^m will contain only Cosine terms. First the determination of the constant term a_0.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\theta^m d\theta \tag{2.1}$$

which evaluates to

$$a_0 = \frac{1}{2\pi}\left\{\frac{\pi^{(m+1)}}{m+1} - \frac{\left(-\pi^{(m+1)}\right)}{m+1}\right\} \tag{2.2}$$

and since m is even, $(m + 1)$ is odd and therefore (2.2) becomes

$$a_0 = \frac{\pi^m}{m+1} \tag{2.3}$$

Now the determination of a_n

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}\theta^m \cos n\theta d\theta \tag{2.4}$$

Integrating (2.4) by parts yields

$$a_n = \left[\begin{array}{l}\frac{2m\pi^{(m-2)}}{n^2}(-1)^n - \frac{2m(m-1)(m-2)\pi^{(m-4)}}{n^4}(-1)^n + \\ \frac{2m(m-1)(m-2)(m-3)(m-4)\pi^{(m-6)}}{n^6}(-1)^n - \cdots + (-1)^{\left(m/2+1\right)}\frac{2m!}{n^m}(-1)^n\end{array}\right]\sum \cos n\theta \tag{2.5}$$

where n = 1 → ∞

Which becomes

$$a_n = \frac{\pi^{(m-2)}}{(m-1)!}\sum_{n=1}^{\infty}\frac{(-1)^n \cos n\theta}{n^2} - \frac{\pi^{(m-4)}}{(m-3)!}\sum_{n=1}^{\infty}\frac{(-1)^n \cos n\theta}{n^4} + \cdots + (-1)^{\left(\frac{m}{2}+1\right)}\sum_{n=1}^{\infty}\frac{(-1)^n \cos n\theta}{n^m} \tag{2.6}$$

so that with (2.3) this yields after some reduction

$$\sum_{n=1}^{\infty}(-1)^n \frac{\cos n\theta}{n^m} = (-1)^{\left(\frac{m}{2}+1\right)}\frac{1}{2m!}\left(\theta^m - \frac{\pi^m}{m+1}\right) + \sum_{x=1}^{\left(\frac{m}{2}-1\right)}\frac{\pi^{(m-2x)}(-1)^{\left(\frac{m}{2}+x+1\right)}}{(m-2x+1)!}(-1)^n \frac{\cos n\theta}{n^{2x}} \tag{2.7}$$

This is the equation from which generalised results for $\zeta(m)$ and $\eta(m)$, m even, can be derived.

First for $\zeta(m)$, in (2.7) put $\theta = \pi$. This gives

$$\sum_{n=1}^{\infty}\frac{1}{n^m} = (-1)^{\left(\frac{m}{2}+1\right)}\frac{\pi^m}{2m!}\left(1 - \frac{1}{m-1}\right) + \sum_{x=1}^{\left(\frac{m}{2}-1\right)}\frac{\pi^{(m-2x)}(-1)^{\left(\frac{m}{2}+x+1\right)}}{(m-2x+1)!}\sum_{n=1}^{\infty}\frac{1}{n^{2x}} \tag{2.8}$$

which clearly becomes

$$\varsigma(m) = (-1)^{\left(\frac{m}{2}+1\right)}\frac{\pi^m}{2(m+1)(m-1)!} + \sum_{x=1}^{\left(\frac{m}{2}-1\right)}\frac{\pi^{(m-2x)}(-1)^{\left(\frac{m}{2}+x+1\right)}}{(m-2x+1)!}\varsigma(2x) \tag{2.9}$$

Examples of the application of (2.9) are given in the next Section.

<u>2.2 The Eta Function, (*m* Even).</u>

In (2.7) put $\theta = 0$ to give

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^m} = -(-1)^{\left(\frac{m}{2}+1\right)}\frac{\pi^m}{2(m+1)(m)!} + \sum_{x=1}^{\left(\frac{m}{2}-1\right)}\frac{\pi^{(m-2x)}(-1)^{\left(\frac{m}{2}+x+1\right)}}{(m-2x+1)!}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{2x}} \tag{2.10}$$

In (2.10)

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^m} = -1 + \frac{1}{2^m} - \frac{1}{3^m} + \cdots = -\eta(m) \tag{2.11}$$

and

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{2x}} = -1 + \frac{1}{2^{2x}} - \frac{1}{3^{2x}} + \cdots = -\eta(2x) \tag{2.12}$$

So that insertion of (2.12) into (2.11) finally gives

$$\eta(m) = (-1)^{\left(\frac{m}{2}+1\right)}\frac{\pi^m}{2(m+1)!} + \sum_{x=1}^{\left(\frac{m}{2}-1\right)}\frac{\pi^{(m-2x)}(-1)^{\left(\frac{m}{2}+x+1\right)}}{(m-2x+1)!}\eta(2x) \tag{2.13}$$

Examples of the application of (2.13) are given in the next Section.

2.3 The Xi Function.

To generalise the closed form of the Xi function, it is necessary to develop a Fourier series expansion of θ^m where m is an odd natural number. In this case θ^m is an odd function and therefore the series expansion will contain only Sine terms. The result of the expansion is

$$\sum_{n=1}^{\infty}(-1)^n \frac{\sin n\theta}{n^m} = (-1)^{\frac{(m+1)}{2}} \frac{\theta^m}{2m!} + \sum_{x=1}^{(m-1)/2} \frac{\pi^{(m-2x+1)}(-1)^{\left((m-1)/2+x\right)}}{(m-2x+2)!}(-1)^n \frac{\sin n\theta}{n^{(2x-1)}} \tag{2.14}$$

where n = 1 → ∞

Now putting θ = π/2 gives

$$\sum_{n=1}^{\infty}(-1)^n \frac{\sin n\pi/2}{n^m} = (-1)^{\frac{(m+1)}{2}} \frac{\pi^m}{2^{(m+1)}m!} + \sum_{x=1}^{(m-1)/2} \frac{\pi^{(m-2x+1)}(-1)^{\left((m-1)/2+x\right)}}{(m-2x+2)!}\sum_{n=1}^{\infty}(-1)^n \frac{\sin n\pi/2}{n^{(2x-1)}} \tag{2.15}$$

In (2.15)

$$\sum_{n=1}^{\infty}(-1)^n \frac{\sin n\pi/2}{n^m} = -1 + \frac{1}{3^m} - \frac{1}{5^m} + \cdots = -\xi(m)$$

and (2.16)

$$\sum_{n=1}^{\infty}(-1)^n \frac{\sin n\pi/2}{n^{(2x-1)}} = -1 + \frac{1}{3^{(2x-1)}} - \frac{1}{5^{2x-1}} + \cdots = -\xi(2x-1)$$

So that insertion of (2.16) into (2.15) finally gives

$$\xi(m) = -(-1)^{\frac{(m+1)}{2}} \frac{\pi^m}{2^{(m+1)}m!} + \sum_{x=1}^{(m-1)/2} \frac{\pi^{(m-2x+1)}(-1)^{\left((m-1)/2+x\right)}}{(m-2x+2)!}\xi(2x-1) \tag{2.17}$$

Examples of the application of (2.17) are given in the next Section.

3.0 Examples of Application.

To fully demonstrate the versatility of this method, in all three cases examples will be produced up to exponent 10. It should be noted that, of necessity, applicable solutions are produced as a ladder function.

3.1 The Zeta Function.

(i) From (2.9) for m = 2.

$$\varsigma(2) = \frac{\pi^2}{2x1!\,x3} = \frac{\pi^2}{6} \tag{3.1}$$

(ii) Inserting (3.1) into (2.9) for m = 4.

$$\varsigma(4) = -\frac{\pi^4}{2x3!\,x5} + \frac{\pi^2}{3!}\varsigma(2) = -\frac{\pi^4}{60} + \frac{\pi^4}{36} = \frac{\pi^4}{90} \tag{3.2}$$

(iii) Inserting (3.1) and (3.2) into (2.9) for *m* = 6.

$$\varsigma(6)=\frac{\pi^6}{2x5!x7}-\frac{\pi^4}{5!}\varsigma(2)+\frac{\pi^2}{3!}\varsigma(4)=\frac{\pi^6}{1680}-\frac{\pi^6}{720}+\frac{\pi^6}{540}=\frac{\pi^6}{945} \tag{3.3}$$

(iv) Inserting (3.1), (3.2) and (3.3) into (2.9) for *m* = 8.

$$\begin{aligned}\varsigma(8)&=-\frac{\pi^8}{2x7!x9}+\frac{\pi^6}{7!}\varsigma(2)-\frac{\pi^4}{5!}\varsigma(4)+\frac{\pi^2}{3!}\varsigma(6)\\&=\frac{\pi^8}{90720}\left(-1+3-8\frac{2}{5}+16\right)=\frac{\pi^8}{9450}\end{aligned} \tag{3.4}$$

(v) Finally, Inserting (3.1), (3.2) ,(3.3) and (3.4) into (2.9) for *m* = 10.

$$\begin{aligned}\varsigma(10)&=\frac{\pi^{10}}{2x9!x11}-\frac{\pi^8}{9!}\varsigma(2)+\frac{\pi^6}{7!}\varsigma(4)+\frac{\pi^4}{5!}\varsigma(6)-\frac{\pi^2}{3!}\varsigma(8)\\&=\frac{\pi^{10}}{22x9!}\left(1-3\frac{2}{3}+17\frac{3}{5}-70\frac{2}{5}+140\frac{4}{5}\right)=\frac{\pi^{10}}{93555}\end{aligned} \tag{3.5}$$

3.2 The Eta Function.

(i) From (2.13) for *m* = 2.

$$\eta(2)=\frac{\pi^2}{2x2!x3}=\frac{\pi^2}{12} \tag{3.6}$$

(ii) Insertion of (3.6) into (2.13) for *m* = 4.

$$\eta(4)=\frac{\pi^4}{2x4!x5}+\frac{\pi^2}{3!}\eta(2)=\frac{7\pi^4}{720} \tag{3.7}$$

(iii) Insertion of (3.6) and (3.7) into (2.13) for *m* = 6.

$$\begin{aligned}\eta(6)&=\frac{\pi^6}{2x6!x7}-\frac{\pi^4}{5!}\eta(2)+\frac{\pi^2}{3!}\eta(4)\\&=\frac{\pi^6}{10{,}080}\left(1-7+16\frac{1}{3}\right)=\frac{31\pi^6}{30{,}240}\end{aligned} \tag{3.8}$$

(iv) Insertion of (3.6), (3.7) and (3.8) into (2.13) for *m* = 8.

$$\begin{aligned}\eta(8)&=-\frac{\pi^8}{2x8!x9}+\frac{\pi^6}{7!}\eta(2)-\frac{\pi^4}{5!}\eta(4)+\frac{\pi^2}{3!}\eta(6)\\&=\frac{\pi^8}{725{,}760}\left(-1+12-58\frac{4}{5}+124\right)=\frac{127\pi^8}{1{,}209{,}600}\end{aligned} \tag{3.9}$$

(v) Insertion of (3.6), (3.7), (3.8) and (3.9) into (2.13) for *m* = 10.

$$\eta(10)=\frac{\pi^{10}}{2x10!\,x11}-\frac{\pi^8}{9!}\eta(2)+\frac{\pi^6}{7!}\eta(4)-\frac{\pi^4}{5!}\eta(6)+\frac{\pi^2}{3!}\eta(8)$$
$$=\frac{\pi^{10}}{22x10!}\left(1-18\frac{1}{3}+154-682+1397\right)=\frac{73\pi^{10}}{6{,}842{,}880} \quad (3.10)$$

3.3 The Xi Function.

(i) From (2.17) for *m* = 1.

$$\xi(1)=\frac{\pi}{2^2}=\frac{\pi}{4} \quad (3.11)$$

(ii) Insertion of (3.11) into (2.17) for *m* = 3.

$$\xi(3)=-\frac{\pi^3}{2^4x3!}+\frac{\pi^2}{3!}\xi(1)=\frac{\pi^3}{32} \quad (3.12)$$

(iii) Insertion of (3.11) and (3.12) into (2.17) for *m* = 5.

$$\xi(5)=\frac{\pi^5}{2^6x5!}-\frac{\pi^4}{5!}\xi(1)+\frac{\pi^2}{3!}\xi(3)$$
$$=\frac{\pi^5}{7{,}680}(1-16+40)=\frac{5\pi^5}{1536} \quad (3.13)$$

(iv) Insertion of (3.11), (3.12) and (3.13) into (2.17) for *m* = 7.

$$\xi(7)=-\frac{\pi^7}{2^8x7!}+\frac{\pi^6}{7!}\xi(1)-\frac{\pi^4}{5!}\xi(3)+\frac{\pi^2}{3!}\xi(5)$$
$$=\frac{\pi^7}{1{,}290{,}240}(-1+64-336+700)=\frac{61\pi^7}{184{,}320} \quad (3.14)$$

(v) Insertion of (3.11), (3.12), (3.13) and (3.14) into (2.17) for *m* = 9.

$$\xi(9)=\frac{\pi^9}{2^{10}x9!}-\frac{\pi^8}{9!}\xi(1)+\frac{\pi^6}{7!}\xi(3)-\frac{\pi^4}{5!}\xi(5)+\frac{\pi^2}{3!}\xi(7)$$
$$=\frac{\pi^9}{2^{10}x9!}(1-256+2304-10080+20496)=\frac{277\pi^9}{8{,}257{,}536} \quad (3.15)$$

4.0 Conclusions.

As stated in the Introduction, the limitations in [1] and [2] have been completely eliminated in the generalisation process presented here. It produces closed form results for all three main infinite series families, and computation for even the largest exponent is relatively short and simple. The only drawback is that in order to determine the closed form of a series for some random exponent *m*, it is necessary to determine the closed form of all preceding exponents, a consequence of the ladder function relationship extant within the series families. This however, is considered only a minor limitation compared to the advantages. The other point to note is that in all the examples shown above, all of the coefficients of π^m are prime numbers, (in many cases this includes unity, which is considered to be a prime number). Because all numbers can be reduced to multiples of primes it would be expected that this feature would apply to the results for all series for all values of m.

REFERENCES.

[1] P.G.Bass, *The Closed Form of Convergent Infinite Series - 1 An Extension of Leonhard Euler's First Method*, www.relativedomains.com.

[2] P.G.Bass, *The Closed Form of Convergent Infinite Series - 2 An Extension of Leonhard Euler's Second Method*, www.relativedomains.com.