# THE CLOSED FORMS OF CONVERGENT INFINITE SERIES

# 2

# AN EXTENSION OF LEONHARD EULER'S SECOND METHOD.

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## Abstract

Leonhard Euler's second method of solving "The Basel Problem" is herein extended to provide closed form solutions to the Zeta function for all even values of the exponent.

# Contents.

## 1.0 Introduction.

Leonhard Euler's second method of the solution of "The Basel Problem" is his most well known, but also reportedly the least rigorous. It nevertheless produces the correct solution. The method is similar to his first in that it compares the terms of (*Sin x*)/*x* as represented first by a M$^c$Claurin series expansion and secondly as a continued product of roots.

In this paper, this method is extended so as to produce closed form results for the Zeta function for all even exponents that conform to a simple criteria. The extension of the method is initially demonstrated via three examples, the first of which is Euler's solution of "The Basel Problem". The second and third demonstrate the extension for exponents 4 and 6. This is then followed by a comprehensive list of solutions for exponents 8 to 20 together with the means to effect solutions for exponents up to 272. The listing is arranged in Groups and Sub-Groups according to certain criteria.

The extension of Euler's first method is the subject of [1].

## 2.0 Extension of Euler's Second Method.

## 2.1 Mathematical Derivation Via Three Examples.

### 2.1.1. Euler's Solution to The Basel Problem.

The Zeta function is given by the infinite series

$$\varsigma(m) = 1 + \frac{1}{2^m} + \frac{1}{3^m} + \frac{1}{4^m} + \frac{1}{5^m} + \cdots \qquad (2.1)$$

where *m* is any even natural number.

"The Basel Problem" solution is the closed form of (2.1) when *m* = 2. To solve this problem, Euler proceeded as follows.

A M$^c$Claurin series expansion of *Sin*(*s*) is

$$\boldsymbol{sin}(s) = s - \frac{s^3}{3!} + \frac{s^5}{5!} - \frac{s^7}{7!} + \frac{s^9}{9!} - \cdots \qquad (2.2)$$

So that

$$\frac{\boldsymbol{sin}(s)}{s} = 1 - \frac{s^2}{3!} + \frac{s^4}{5!} - \frac{s^6}{7!} + \frac{s^8}{9!} - \cdots \qquad (2.3)$$

The roots of *Sin*(*s*) occur when *s* = ... -2π, -π, π, 2π ... etc so that *Sin*(*s*)/*s* may be represented by the infinite product

$$\frac{\boldsymbol{sin}(s)}{s} = \left(1 - \frac{s}{\pi}\right)\left(1 + \frac{s}{\pi}\right)\left(1 - \frac{s}{2\pi}\right)\left(1 + \frac{s}{2\pi}\right)\cdots \qquad (2.4)$$

Which reduces to

$$\frac{\boldsymbol{sin}(s)}{s} = \left(1 - \frac{s^2}{\pi^2}\right)\left(1 - \frac{s^2}{(2\pi)^2}\right)\cdots \qquad (2.5)$$

Multiplying out (2.5) taking just the squared terms and matching them with the squared term in (2.3) gives

$$-\frac{s^2}{\pi^2} - \frac{s^2}{(2\pi)^2} - \frac{s^2}{(3\pi)^2} - \cdots = -\frac{s^2}{3!} \qquad (2.6)$$

So that

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{6} \qquad (2.7)$$

This solution is much simpler than Euler's first method but only deals with exponent $m$ = 2.

Note that in the remainder of this paper, terms such as $Sin(s)$ et al will be referred to as the Describing Term and equations such as (2.4) as the Root Equation.
A comprehensive explanation of Euler's second method of the solution of "The Basel Problem is given in [2].

### 2.1.2. Extension of Euler's Second Method to *m* = 4.

Consider the Describing Term

$$f(s)=\boldsymbol{sin}(s)\boldsymbol{sin}(js) \qquad (2.8)$$

A McClaurin series expansion of (2.8) is

$$\boldsymbol{sin}(s)\boldsymbol{sin}(js)=js^2-\frac{8js^6}{6!}+\cdots \qquad (2.9)$$

so that

$$\frac{\boldsymbol{sin}(s)\boldsymbol{sin}(js)}{js^2}=1-\frac{8s^4}{6!}+\cdots \qquad (2.10)$$

The Root Equation comparable to (2.10) is

$$\frac{\boldsymbol{sin}(s)\boldsymbol{sin}(js)}{js^2}=\left(1-\frac{s}{\pi}\right)\left(1+\frac{s}{\pi}\right)\left(1-\frac{js}{\pi}\right)\left(1+\frac{js}{\pi}\right)\cdots \qquad (2.11)$$

which reduces to

$$\begin{aligned}\frac{\boldsymbol{sin}(s)\boldsymbol{sin}(js)}{js^2}&=\left(1-\frac{s^2}{\pi^2}\right)\left(1+\frac{s^2}{\pi^2}\right)\left(1-\frac{s^2}{(2\pi)^2}\right)\left(1+\frac{s^2}{(2\pi)^2}\right)\cdots\\&=\left(1-\frac{s^4}{\pi^4}\right)\left(1-\frac{s^4}{(2\pi)^4}\right)\cdots\end{aligned} \qquad (2.12)$$

Multiplying out (2.12) taking just terms in $s^4$ and matching them to the term in $s^4$ in (2.10) gives

$$-\frac{s^4}{\pi^4}-\frac{s^4}{(2\pi)^4}-\frac{s^4}{(3\pi)^4}-\cdots=-\frac{8s^4}{6!} \qquad (2.13)$$

so that

$$1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\cdots=\frac{8\pi^4}{6!}=\frac{\pi^4}{90} \tag{2.14}$$

### 2.1.3. **Extension of Euler's Second Method to *m* = 6.**

For exponent $m = 6$, it is clear from the above two examples that the Root Equation will be

$$R(s)=\left(1-\frac{s^6}{\pi^6}\right)\left(1-\frac{s^6}{(2\pi)^6}\right)\left(1-\frac{s^6}{(3\pi)^6}\right)\cdots \tag{2.15}$$

Consider just the first term in (2.15), (the other terms will follow an identical pattern),

$$R_1(s)=\left(1-\frac{s^6}{\pi^6}\right)=\left(1-\frac{s^3}{\pi^3}\right)\left(1+\frac{s^3}{\pi^3}\right) \tag{2.16}$$

It is not possible to generate a Describing Term for (2.16) because of the cubic terms, (exponents must always be even). Consequently it is necessary to suitably modify the Root Equation. Thus consider the following first term of the modified Root Equation.

$$R_1(s)=\left(1-\frac{s^2}{\pi^2}\right)^2\left(1+\frac{2s^2}{\pi^2}\right) \tag{2.17}$$

When multiplied out this becomes

$$R_1(s)=\left(1-\frac{3s^4}{\pi^4}+\frac{2s^6}{\pi^6}\right) \tag{2.18}$$

and including the other terms the full Root Equation becomes

$$R(s)=\left(1-\frac{3s^4}{\pi^4}+\frac{2s^6}{\pi^6}\right)\left(1-\frac{3s^4}{(2\pi)^4}+\frac{2s^6}{(2\pi)^6}\right)\left(1-\frac{3s^4}{(3\pi)^4}+\frac{2s^6}{(3\pi)^6}\right)\cdots \tag{2.19}$$

It is important to note that in (2.17) multiplication by the hyperbolic root removes the $s^2$ term in (2.18). This is necessary because in the Root Equation exponents must not be multiples of each other otherwise erroneous solutions would result. From (2.17) it is clear that the Describing Term is

$$f(s)=j\,\boldsymbol{sin}^2(s)\boldsymbol{sinh}\left(\sqrt{2}s\right) \tag{2.20}$$

A M^c^Claurin series expansion of (2.20) yields

$$j\,\boldsymbol{sin}^2(s)\boldsymbol{sinh}\left(\sqrt{2}s\right)=j\sqrt{2}s^3-\frac{j168\sqrt{2}s^7}{7!}+\frac{j768\sqrt{2}s^9}{9!}-\cdots \tag{2.21}$$

so that

$$\frac{\boldsymbol{sin}^2(s)\boldsymbol{sinh}\left(\sqrt{2}s\right)}{\sqrt{2}s^3}=1-\frac{168s^4}{7!}+\frac{768s^6}{9!}-\cdots \tag{2.22}$$

Now multiplying out (2.19) taking only terms in $s^4$ and $s^6$ and matching them with like terms in (2.22) gives

$$-\frac{3s^4}{\pi^4}-\frac{3s^4}{(2\pi)^4}-\frac{3s^4}{(3\pi)^4}-\cdots=-\frac{168s^4}{7!} \qquad (2.23)$$

giving

$$1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}\cdots=\frac{56\pi^4}{7!}=\frac{\pi^4}{90} \qquad (2.24)$$

and

$$\frac{2s^6}{\pi^6}+\frac{2s^6}{(2\pi)^6}+\frac{2s^6}{(3\pi)^6}+\cdots=\frac{768s^6}{9!} \qquad (2.25)$$

so that

$$1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}\cdots=\frac{384\pi^6}{9!}=\frac{\pi^6}{945} \qquad (2.26)$$

Note that the modified Root Equation has yield results for $m$ = 4 and $m$ = 6.

To proceed further it is only necessary to multiply (2.17) by $\left(1+\frac{3s^4}{\pi^4}\right)$ to produce a Root equation 1st term of

$$R_1(s)=1+\frac{2s^6}{\pi^6}-\frac{9s^8}{\pi^8}+\frac{6s^{10}}{\pi^{10}} \qquad (2.27)$$

in which the term in $s^4$ has been removed, (factor of $s^8$ ), thus resulting in solutions for $m$ = 6, 8 and 10. The Describing Term for this example would be, from (2.17) as multiplied by $\left(1+\frac{3s^4}{\pi^4}\right)$

$$f(s)=-\boldsymbol{sin}^2(s)\boldsymbol{sinh}\left(\sqrt{2}s\right)\boldsymbol{sin}\left\{(-3)^{1/4}s\right\}\boldsymbol{sinh}\left\{(-3)^{1/4}s\right\} \qquad (2.28)$$

In obtaining a McClaurin series expansion for this Describing Term it is easier to obtain the expansions for each separate term and then multiply, (the degree of computation is significantly less).
In proceeding further, the tables of the next section are produced.

## 2.2 Tables of the Root Equations, Describing Terms, Exponents Covered and Closed Form Results.

To facilitate categorisation, these tables are presented in order of a Group Number incorporating four, (of an infinitude), of Sub-groups.

| **Sub-Group** | **Function** | **Expansion** | **Main Exp't** | **Sec'y Exp'ts** | **Main Exp't Result** |
|---|---|---|---|---|---|
| 1 | Root Eqn. | $\left(1-\frac{s^2}{\pi^2}\right)\left(1-\frac{s^2}{(2\pi)^2}\right)\left(1-\frac{s^2}{(3\pi)^2}\right)\cdots$ | 2 | | $\frac{\pi^2}{3!}=\frac{\pi^2}{6}$ |
| | 1st Root Term | $\left(1-\frac{s^2}{\pi^2}\right)$ | | | |
| | Describing Term | $sin(s)$ | | | |
| 2 | Root Eqn. | $\left(1-\frac{s^4}{\pi^4}\right)\left(1-\frac{s^4}{(2\pi)^4}\right)\left(1-\frac{s^4}{(3\pi)^4}\right)\cdots$ | 4 | | $\frac{8\pi^2}{6!}=\frac{\pi^4}{90}$ |
| | 1st Root Term | $\left(1-\frac{s^2}{\pi^2}\right)\left(1+\frac{s^2}{\pi^2}\right)$ | | | |
| | Describing Term | $jsin(s)sinh(s)$ | | | |
| 3 | Root Eqn. | $\left(1-\frac{s^8}{\pi^8}\right)\left(1-\frac{s^8}{(2\pi)^8}\right)\left(1-\frac{s^8}{(3\pi)^8}\right)\cdots$ | 8 | | $\frac{384\pi^8}{10!}=\frac{\pi^8}{9450}$ |
| | 1st Root Term | $\left(1-\frac{s^2}{\pi^2}\right)\left(1+\frac{s^2}{\pi^2}\right)\left(1-\frac{js^2}{\pi^2}\right)\left(1+\frac{js^2}{\pi^2}\right)$ | | | |
| | Describing Term | $-sin(s)sinh(s)sin\{(j)^{1/2}s\}sinh\{(j)^{1/2}s\}$ | | | |
| 4 | Root Eqn. | $\left(1-\frac{s^{16}}{\pi^{16}}\right)\left(1-\frac{s^{16}}{(2\pi)^{16}}\right)\left(1-\frac{s^{16}}{(3\pi)^{16}}\right)\cdots$ | 16 | | $\frac{27{,}022{,}983{,}168\pi^{16}}{20!}$ |
| | 1st Root Term | $\left(1-\frac{s^2}{\pi^2}\right)\left(1+\frac{s^2}{\pi^2}\right)\left(1-\frac{js^2}{\pi^2}\right)\left(1+\frac{js^2}{\pi^2}\right)\left(1-\frac{(-j)^{1/2}s^2}{\pi^2}\right)\left(1+\frac{(-j)^{1/2}s^2}{\pi^2}\right)\left(1-\frac{(j)^{1/2}s^2}{\pi^2}\right)\left(1+\frac{(j)^{1/2}s^2}{\pi^2}\right)$ | | | |
| | Describing Term | $sin(s)sinh(s)sin\{(j)^{1/2}s\}sinh\{(j)^{1/2}s\}sin\{(-j)^{1/4}s\}sinh\{(-j)^{1/4}s\}\ sin\{(j)^{1/4}s\}sinh\{(j)^{1/4}s\}$ | | | |

etc

**Table 2.1 - Group 1 Function Root Equations, Describing Terms, Exponents Covered and Closed Form Results.**

| Sub-Group | Function | Expansion | Main Exp't | Sec'y Exp'ts | Main Exp't Result |
|---|---|---|---|---|---|
| 1 | Root Eqn. | $\left(1-\frac{s^2}{\pi^2}\right)^2\left(1+\frac{2s^2}{\pi^2}\right)\left(1-\frac{s^2}{(2\pi)^2}\right)^2\left(1+\frac{2s^2}{(2\pi)^2}\right)\left(1-\frac{s^2}{(3\pi)^2}\right)^2\left(1+\frac{2s^2}{(3\pi)^2}\right)\cdots$ | 6 | 4 | $\frac{384\pi^6}{9!}=\frac{\pi^6}{945}$ |
| | 1st Root Term | $\left(1-\frac{3s^4}{\pi^4}+\frac{2s^6}{\pi^6}\right)$ | | | |
| | Describing Term | $jsin(s)sinh\{(2)^{1/2}s\}$ | | | |
| 2 | Root Eqn. | $\left(1-\frac{s^4}{\pi^4}\right)^2\left(1+\frac{s^4}{\pi^4}\right)\left(1-\frac{s^4}{(2\pi)^4}\right)^2\left(1+\frac{s^4}{(2\pi)^4}\right)\left(1-\frac{s^4}{(3\pi)^4}\right)^2\left(1+\frac{s^4}{(3\pi)^4}\right)\cdots$ | 12 | 8 | $\frac{1,415,168\pi^{12}}{15!}$ |
| | 1st Root Term | $\left(1-\frac{3s^8}{\pi^8}+\frac{2s^{12}}{\pi^{12}}\right)$ | | | |
| | Describing Term | $jsin^2(s)sinh^2(s)sin\{(-2)^{1/4}s\}sinh\{(-2)^{1/4}s\}$ | | | |
| 3 | Root Eqn. | $\left(1-\frac{s^8}{\pi^8}\right)^2\left(1+\frac{2s^8}{\pi^8}\right)\left(1-\frac{s^8}{(2\pi)^8}\right)^2\left(1+\frac{2s^8}{(2\pi)^8}\right)\left(1-\frac{s^8}{(3\pi)^8}\right)^2\left(1+\frac{2s^8}{(3\pi)^8}\right)\cdots$ | 24 | 16 | Not Evaluated |
| | 1st Root Term | $\left(1-\frac{3s^{16}}{\pi^{16}}+\frac{2s^{24}}{\pi^{24}}\right)$ | | | |
| | Describing Term | 8 circular functions required | | | |
| 4 | Root Eqn. | $\left(1-\frac{s^{16}}{\pi^{16}}\right)^2\left(1+\frac{2s^{16}}{\pi^{16}}\right)\left(1-\frac{s^{16}}{(2\pi)^{16}}\right)^2\left(1+\frac{2s^{16}}{(2\pi)^{16}}\right)\left(1-\frac{s^{16}}{(3\pi)^{16}}\right)^2\left(1+\frac{2s^{16}}{(3\pi)^{16}}\right)\cdots$ | 48 | 32 | Not Evaluated |
| | 1st Root Term | $\left(1-\frac{3s^{32}}{\pi^{32}}+\frac{2s^{48}}{\pi^{48}}\right)$ | | | |
| | Describing Term | 16 circular functions required | | | |

etc

**Table 2.2 - Group 2 Function Root Equations, Describing Terms, Exponents Covered and Closed Form Results.**

| Sub-Group | Function | Expansion | Main Exp't | Sec'y Exp'ts | Main Exp't Result |
|---|---|---|---|---|---|
| 1 | Root Eqn. | $\left(1-\frac{s^2}{\pi^2}\right)^2\left(1+\frac{2s^2}{\pi^2}\right)\left(1+\frac{3s^4}{\pi^4}\right)\left(1-\frac{s^2}{(2\pi)^2}\right)^2\left(1+\frac{2s^2}{(2\pi)^2}\right)\left(1+\frac{3s^4}{(2\pi)^4}\right)\cdots$ | 10 | 6 & 8 | $\frac{5120\pi^{10}}{12!}=\frac{\pi^{10}}{93555}$ |
| | 1st Root Term | $\left(1+\frac{2s^6}{\pi^6}-\frac{9s^8}{\pi^8}+\frac{6s^{10}}{\pi^{10}}\right)$ | | | |
| | Describing Term | $-\sin^2(s)\sinh\{(2)^{1/2}s\}\sin\{(-3)^{1/4}s\}\sinh\{(-3)^{1/4}s\}$ | | | |
| 2 | Root Eqn. | $\left(1-\frac{s^4}{\pi^2}\right)^2\left(1+\frac{2s^4}{\pi^2}\right)\left(1+\frac{3s^8}{\pi^8}\right)\left(1-\frac{s^4}{(2\pi)^4}\right)^2\left(1+\frac{2s^4}{(2\pi)^4}\right)\left(1+\frac{3s^8}{(2\pi)^8}\right)\cdots$ | 20 | 12 & 16 | $\frac{1{,}768{,}677{,}452{,}021{,}760\pi^{20}}{22!}$ |
| | 1st Root Term | $\left(1+\frac{2s^{12}}{\pi^{12}}-\frac{9s^{16}}{\pi^{16}}+\frac{6s^{20}}{\pi^{20}}\right)$ | | | |
| | Describing Term | 8 circular functions required. | | | |
| 3 | Root Eqn. | $\left(1-\frac{s^8}{\pi^8}\right)^2\left(1+\frac{2s^8}{\pi^8}\right)\left(1+\frac{3s^{16}}{\pi^{16}}\right)\left(1-\frac{s^8}{(2\pi)^8}\right)^2\left(1+\frac{2s^8}{(2\pi)^8}\right)\left(1+\frac{3s^{16}}{(2\pi)^{16}}\right)\cdots$ | 40 | 24 & 32 | Not Evaluated |
| | 1st Root Term | $\left(1+\frac{2s^{24}}{\pi^{24}}-\frac{9s^{32}}{\pi^{32}}+\frac{6s^{40}}{\pi^{40}}\right)$ | | | |
| | Describing Term | 16 circular functions required | | | |
| 4 | Root Eqn. | $\left(1-\frac{s^{16}}{\pi^{16}}\right)^2\left(1+\frac{2s^{16}}{\pi^{16}}\right)\left(1+\frac{3s^{32}}{\pi^{32}}\right)\left(1-\frac{s^{16}}{(2\pi)^{16}}\right)^2\left(1+\frac{2s^{16}}{(2\pi)^{16}}\right)^2\left(1+\frac{3s^{32}}{(2\pi)^{32}}\right)\cdots$ | 80 | 48 & 64 | Not Evaluated |
| | 1st Root Term | $\left(1+\frac{2s^{48}}{\pi^{48}}-\frac{9s^{64}}{\pi^{64}}+\frac{6s^{80}}{\pi^{80}}\right)$ | | | |
| | Describing Term | 32 circular functions required | | | |

etc

**Table 2.3 - Group 3 Function Root Equations, Describing Terms, Exponents Covered and Closed Form Results.**

| Sub-Group | Function | Expansion | Main Exp't | Sec'y Exp'ts | Main Exp't Result |
|---|---|---|---|---|---|
| 1 | Root Eqn. | $\left(1-\frac{s^2}{\pi^2}\right)^2\left(1+\frac{2s^2}{\pi^2}\right)\left(1+\frac{3s^4}{\pi^4}\right)\left(1+\frac{9s^8}{\pi^8}\right)\left(1-\frac{s^2}{(2\pi)^2}\right)^2\left(1+\frac{2s^2}{(2\pi)^2}\right)\left(1+\frac{3s^4}{(2\pi)^4}\right)\left(1+\frac{9s^8}{(2\pi)^8}\right)\cdots$ | 14 | 6 & 10 | $\frac{2{,}293{,}760\pi^{14}}{16!}$ |
| | 1st Root Term | $\left(1+\frac{2s^6}{\pi^6}+\frac{6s^{10}}{\pi^{10}}+\frac{18s^{14}}{\pi^{14}}-\frac{81s^{16}}{\pi^{16}}+\frac{54s^{18}}{\pi^{18}}\right)$ | | | |
| | Describing Term | $-j\sin^2(s)\sinh\{(2)^{1/2}s\}\sin\{(-3)^{1/4}s\}\sinh\{(-3)^{1/4}s\}\sin\{(-j3)^{1/4}s\}\ \sinh\{(-j3)^{1/4}s\}\sin\{(j3)^{1/4}s\}$ $\sinh\{(j3)^{1/4}s\}$ | | | |
| 2 | Root Eqn. | $\left(1-\frac{s^4}{\pi^4}\right)^2\left(1+\frac{2s^4}{\pi^4}\right)\left(1+\frac{3s^8}{\pi^8}\right)\left(1+\frac{9s^{16}}{\pi^{16}}\right)\left(1-\frac{s^4}{(2\pi)^4}\right)^2\left(1+\frac{2s^4}{(2\pi)^4}\right)\left(1+\frac{3s^8}{(2\pi)^8}\right)\left(1+\frac{9s^{16}}{(2\pi)^{16}}\right)\cdots$ | 28 | 12 & 20 | Not Evaluated |
| | 1st Root Term | $\left(1+\frac{2s^{12}}{\pi^{12}}+\frac{6s^{20}}{\pi^{20}}+\frac{18s^{28}}{\pi^{28}}-\frac{81s^{32}}{\pi^{32}}+\frac{54s^{36}}{\pi^{36}}\right)$ | | | |
| | Describing Term | 16 circular functions required. | | | |
| 3 | Root Eqn. | $\left(1-\frac{s^8}{\pi^8}\right)^2\left(1+\frac{2s^8}{\pi^8}\right)\left(1+\frac{3s^{16}}{\pi^{16}}\right)\left(1+\frac{9s^{32}}{\pi^{32}}\right)\left(1-\frac{s^8}{(2\pi)^8}\right)^2\left(1+\frac{2s^8}{(2\pi)^8}\right)\left(1+\frac{3s^{16}}{(2\pi)^{16}}\right)\left(1+\frac{9s^{32}}{(2\pi)^{32}}\right)\cdots$ | 56 | 24 & 40 | Not Evaluated |
| | 1st Root Term | $\left(1+\frac{2s^{24}}{\pi^{24}}+\frac{6s^{40}}{\pi^{40}}+\frac{18s^{56}}{\pi^{56}}-\frac{81s^{64}}{\pi^{64}}+\frac{54s^{72}}{\pi^{72}}\right)$ | | | |
| | Describing Term | 32 circular functions required | | | |
| 4 | Root Eqn. | $\left(1-\frac{s^{16}}{\pi^{16}}\right)^2\left(1+\frac{2s^{16}}{\pi^{16}}\right)\left(1+\frac{3s^{32}}{\pi^{32}}\right)\left(1+\frac{9s^{64}}{\pi^{64}}\right)\left(1-\frac{s^{16}}{(2\pi)^{16}}\right)^2\left(1+\frac{2s^{16}}{(2\pi)^{16}}\right)\left(1+\frac{3s^{32}}{(2\pi)^{32}}\right)\left(1+\frac{9s^{64}}{(2\pi)^{64}}\right)\cdots$ | 112 | 48 & 80 | Not Evaluated |
| | 1st Root Term | $\left(1+\frac{2s^{48}}{\pi^{48}}+\frac{6s^{80}}{\pi^{80}}+\frac{18s^{112}}{\pi^{112}}-\frac{81s^{128}}{\pi^{128}}+\frac{54s^{144}}{\pi^{144}}\right)$ | | | |
| | Describing Term | 64 circular functions required | | | |

etc

**Table 2.4 - Group 4 Function Root Equations, Describing Terms, Exponents Covered and Closed Form Results.**

Note that in the above tables the following identities are apparent.

(i) In Table 2.1 the main exponents covered in Group 1 conform to the relationship $m = 1 \times 2^n$ where $n$ is the Sub-Group number.

(ii) In Table 2.2 the main exponents covered in Group 2 conform to the relationship $m = 3 \times 2^n$ and the secondary exponents to $2^{(n+1)}$ where $n$ is the Sub-Group number.

(iii) In Table 2.3 the main exponents covered in Group 3 conform to the relationship $m = 5 \times 2^n$ and the secondary exponents to $3 \times 2^n$ and $2^{(n+2)}$ where $n$ is the Sub-Group number.

(iv) In Table 2.4 the main exponents covered in Group 4 conform to the relationship $m = 7 \times 2^n$ and the secondary exponents to $3 \times 2^n$ and $5 \times 2^n$ where $n$ is the Sub-Group number.

**Group 5.**

Group 5 equations are obtained by eliminating the term $81\left(s/\pi\right)^{2^q}$ where q = 4, 5, 6, 7 etc from Group 4 equations. It thereby provides closed forms for the main exponents $m$ = 22 to 272, ($m$ =11x$2^n$, 13x$2^n$, 17x$2^n$) and secondary exponents m = 6 to 112, ($m$ = 3x$2^n$, 5x$2^n$, 7x$2^n$). The Group requires from 16 to 128 circular functions according to Sub-Group.
Clearly, for each succeeding Group, the closed forms of an increasing number of both main and secondary exponents are revealed. However, there are exceptions as discussed in the next Section.

## 2.3 Exponent Exceptions and Generalisation.

### 2.3.1. Exceptions.

From the above results it is clear that the exponents covered by this extension of Euler's second method are given by the following identity

$$m = p2^n \tag{2.29}$$

where $p$ is a prime number. However, because of this there are some exponents that are not covered. These exponents are given by

$$m = q2^n \tag{2.30}$$

where $q$ is an odd number that is not prime. Within (2.29) and (2.30) the number 2 is now referred to as the Base Number.
Within the first 100 natural numbers there are 15 exponents that do not conform to (2.29) and therefore do conform to (2.30).They are tabulated below,

| Exponent $m$ | $q2^m$ | $pB^n$ |
|---|---|---|
| 18 | 9 x 2 | 3 x 6 |
| 30 | 15 x 2 | 5 x 6 |
| 36 | 9 x $2^2$ | 1 x $6^2$ |
| 42 | 21 x 2 | 7 x 6 |
| 50 | 25 x 2 | 5 x 10 |
| 54 | 27 x 2 | 3 x 18 |
| 60 | 15 x $2^2$ | 5 x 12 |

| | | |
|---|---|---|
| 66 | 33 x 2 | 11 x 6 |
| 70 | 35 x 2 | 7 x 10 |
| 72 | 9 x $2^3$ | 3 x 24 |
| 78 | 39 x 2 | 13 x 6 |
| 84 | 21 x $2^2$ | 7 x 12 |
| 90 | 45 x 2 | 5 x 18 |
| 98 | 49 x 2 | 7 x 14 |
| 100 | 25 x $2^2$ | 1 x $10^2$ |

**Table 2.5 - Exponents Not Covered Using Base Number 2.**

The third column in Table 2.5 shows which Base Number would be required to determine the closed forms of the series for the applicable *m*. As an example of using a Base Number different from 2, Appendix A shows the use of 6 as the Base Number for the determination of the closed form for the series with exponent 18, i.e. $m = 3 \times 6^1$.

### 2.3.2. Generalisation.

To find the closed form using this method for any exponent *m*, follow the procedure below to determine the Base Number and Root Equation.

(i) Divide *m* by each odd prime starting with unity. The closed form is solvable when

$$m = pB^n \qquad (2.31)$$

where, in order

*B* is the lowest Base Number

*n* is the lowest Sub-Group Number

*p* is the lowest prime factor

(ii) The Group Number, *G*, is then given by

$$2^{(G-1)} < 2(p-1) \le 2^G \qquad (2.32)$$

except when $2(p - 1) = 0$ when $G = 1$.

(iii) The Root Equation is then given by

(a) If $G = 1$

$$R_1(s) = \left\{1 - \left(\frac{s}{\pi}\right)^{B^n}\right\} \qquad (2.33)$$

(b) If $G = 2$

$$R_1(s) = \left\{1 - \left(\frac{s}{\pi}\right)^{B^n}\right\}^2 \left\{1 + 2\left(\frac{s}{\pi}\right)^{B^n}\right\} \qquad (2.34)$$

(c) If $G > 2$ $$R_1(s) = \left\{1 - \left(\frac{s}{\pi}\right)^{B^n}\right\}^2 \left\{1 + 2\left(\frac{s}{\pi}\right)^{B^n}\right\} \prod_{q=1}^{G-2} \left\{1 + 3^{2^{(q-1)}} \left(\frac{s}{\pi}\right)^{2^{(q+n)}}\right\} \qquad (2.35)$$

$R_1$(s) is then broken down to multiples of terms of the form $\left\{1 \pm k\left(\frac{s}{\pi}\right)^2\right\}$ from which the Describing Term can be determined. Each component of this is then, via McClaurin, expanded to the level $s^m$, which are then multiplied together so that the final term in $s^m$ can be matched in the full Root equation thereby resulting in the closed form for $\zeta(m)$.

## 3.0 Conclusions.

This extension of Euler's second method of the solution of the "Basel Problem" has concentrated on using Sine as the primary function. It is easy to show, using an identical process, that Cosine could equally well be used.

The practicality of this extension is however somewhat limited. The reasons are threefold. Firstly it only produces results for the Zeta function, (Cosine produces results for the odd terms of the Zeta function). Secondly with increasing values of exponent *m*, the amount of computation required becomes prohibitedly large, and thirdly, not all exponents can be covered by the same Base Number and many different Base Numbers would be required to cover all exponents. Also, with large Base Numbers the amount of computation required becomes even greater.

There are only two positive attributes of the method. The first is that it does allow a choice of which order of Zeta function, (i.e. *m*), is to be solved. Secondly, with increasing *m* an increasing number of multiple results can be obtained. Overall however, it is considered that the negative aspects outweigh the positive, primarily due to the excessive computational burden as *m* and *B* increase.

# Appendix A

## Solution for Exponent 18 Using 6 as the Base Number.

In the main text, Section 2.1.3 it was shown that the closed form for $m$ = 18 et al could not be determined using 2 as the Base Number as it did not conform to the identity of (2.29). It can however, be determined using 6 as the Base Number, i.e. via (2.31),

$$18 = 3 \text{ x } 6^1 \tag{A.1}$$

So that $p$ = 3, $B$ = 6 and $n$ = 1 in which case $G$ would be determined thus

$$2(p-1) = 4$$

So that G must be 2 as from (2.32)

$$2^{(G\text{-}1)} = 2$$

$$2^G = 4$$

$R_1$(s) is then, from (2.34) determined as

$$R_1(s) = \left\{1 - \left(\frac{s}{\pi}\right)^6\right\}^2 \left\{1 + 2\left(\frac{s}{\pi}\right)^6\right\} \tag{A.2}$$

Multiplied out (A.2) becomes

$$R_1(s) = 1 - 3\left(\frac{s}{\pi}\right)^{12} + 2\left(\frac{s}{\pi}\right)^{18} \tag{A.3}$$

So that this will give the closed form results for $m$ = 12 and 18. To determine the Describing Term, the roots of each term in (A.2) must be expressed in terms of $\left\{1 \pm k\left(\frac{s}{\pi}\right)^2\right\}$, thus

$$\begin{aligned} R_1(s) = &\left\{1 + \left(\frac{1}{2} + j\sqrt{\frac{3}{4}}\left(\frac{s}{\pi}\right)^2\right)\right\}^2 \left\{1 + \left(\frac{1}{2} - j\sqrt{\frac{3}{4}}\left(\frac{s}{\pi}\right)^2\right)\right\}^2 \left\{1 - \left(\frac{s}{\pi}\right)^2\right\} \\ &\left\{1 + 1.26\left(\frac{s}{\pi}\right)^2\right\}\left\{1 - (0.63 + j1.09)\left(\frac{s}{\pi}\right)^2\right\}\left\{1 - (0.63 - j1.09)\left(\frac{s}{\pi}\right)^2\right\} \end{aligned} \tag{A.4}$$

and this gives a Describing Term of

$$\begin{aligned} f(s) = &-j\,\boldsymbol{sin}^2(s)\boldsymbol{sinh}^2\left\{\left(\frac{1}{2} + j\sqrt{\frac{3}{4}}\right)(s)\right\}\boldsymbol{sinh}^2\left\{\left(\frac{1}{2} - j\sqrt{\frac{3}{4}}\right)(s)\right\} \\ &\boldsymbol{sinh}\left\{(1.26)^{1/2}(s)\right\}\boldsymbol{sin}\left\{(0.63 + j1.09)^{1/2}(s)\right\}\boldsymbol{sin}\left\{(0.63 - j1.09)^{1/2}(s)\right\} \end{aligned} \tag{A.5}$$

and all terms must be expanded via M$^{c}$Claurin to $s^{18}$ and then multiplied together so that the final term in $s^{18}$ can be matched to that in (A.3). This results in

$$f(s)=1-\frac{4{,}245{,}504s^{12}}{15!}+\frac{114{,}994{,}708{,}480s^{18}}{21!} \qquad \text{(A.6)}$$

Matching terms in (A.6) to those in the full Root Equation of which (A.3) is the first term gives

$$-3\left(\frac{s}{\pi}\right)^{12}-3\left(\frac{s}{2\pi}\right)^{12}-3\left(\frac{s}{3\pi}\right)^{12}\cdots=-\frac{4{,}245{,}504s^{12}}{15!} \qquad \text{(A.7)}$$

so that

$$\varsigma(12)=1+\frac{1}{12^{12}}+\frac{1}{3^{12}}+\cdots=\frac{1{,}415{,}168\pi^{12}}{15!} \qquad \text{(A.8)}$$

and

$$2\left(\frac{s}{\pi}\right)^{18}+2\left(\frac{s}{2\pi}\right)^{18}+2\left(\frac{s}{3\pi}\right)^{18}-\cdots=\frac{114{,}994{,}708{,}480s^{18}}{21!} \qquad \text{(A.9)}$$

which gives

$$\varsigma(18)=1+\frac{1}{12^{18}}+\frac{1}{3^{18}}+\cdots=\frac{57{,}497{,}354{,}240s^{18}}{21!} \qquad \text{(A.10)}$$

## **References**


[1] P.G.Bass, *The Closed Form of Convergent Infinite Series - 1 - An Extension of Leonhard Euler's First Method*, www.relativitydomains.com.

[2] C.Edward Sandifer, *The Early Mathematics of Leonhard Euler*, The Mathematical Association of America, 2007.