THE CLOSED FORMS OF CONVERGENT INFINITE SERIES

1

AN EXTENSION OF LEONHARD EULER'S FIRST METHOD.

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Abstract.

This short paper presents an extension of Leonhard Euler's first method in the determination of the closed forms of convergent infinite series. The extension is to the Cosine and Tangent circular functions.

Contents

1.0 Preface.

This is the first in a series of six papers concerning the closed forms of convergent infinite series, some of which are largely associated with the work of Leonhard Euler. The papers are briefly described below.

(i) This first paper extends Leonhard Euler's first method for the determination of closed forms of such series, with which, using the Sine circular function, he solved "The Basel problem" in 1730. The extension here is to the Cosine and Tangent circular functions.

(ii) The second paper extends Euler's second method, again with which he solved "The Basel Problem". The extension here is to the determination of the closed form of the Zeta Function to all even powers of the exponent.

(iii) The third paper uses the theory of Fourier series expansion to produce three generalised equations that generate the closed forms of all even powers of the Zeta and Eta Functions and all odd powers of the Xi Function.

(iv) The fourth paper introduces a new method for the determination of the closed forms of the Zeta and Eta Functions with even powers and of the Xi function with odd powers. The method uses recursive integration.

(v) For infinite series for which a closed form does not currently exist, the fifth paper introduces a new numerical method for the estimation of their series sum, (~ closed form), to any desired degree of precision using the minimum amount of computation. This particularly applies to the Zeta and Eta Series with odd powers and the Xi series with even powers.

(vi) The sixth and final paper presents some logarithmic integrals and the closed forms of the infinite series they represent. The process is a recursive one so that generalisation is also effected.

2.0 Introduction.

Leonhard Euler's first method for the solution of "The Basel Problem" as shown in [1], was his most productive because it not only provided a solution for $\zeta(2)$, it also provided a solution for $\zeta(m)$ where m is any even Natural Number. In addition it also provided a solution for many other related infinite series. The method is essentially one of comparing values of the Sine function as represented by a M^{c}Claurin series expansion with those of the same function represented by an infinite product. The method is briefly reviewed in Appendix A.

In extending the method to the Cosine and Tangent functions, nothing new is produced here other than the the derivation of the closed forms of a number of new infinite series. In addition the reason for the presence of an anomaly in the use of the classical expansion of the Tangent function is also shown.

3.0 Extension to the Cosine Function.

Prior to the development, it is useful to recollect the nature of the main series involved. They are

$$\begin{aligned}&(i)\quad \textit{The Zeta Function},\ \zeta(m)=1+\frac{1}{2^m}+\frac{1}{3^m}+\frac{1}{4^m}+\cdots\\&(ii)\quad \textit{The Eta Function},\ \eta(m)=1-\frac{1}{2^m}+\frac{1}{3^m}-\frac{1}{4^m}+\cdots\\&(iii)\quad \textit{The Xi Function},\ \xi(m)=1-\frac{1}{3^m}+\frac{1}{5^m}-\frac{1}{7^m}+\cdots\end{aligned}\tag{3.1}$$

where for the Zeta and Eta Functions, *m* is an even Natural Number and for the Xi Function, an odd Natural Number.

Following Euler's method and nomenclature, Cosine is expressed as a M^cClaurin series expansion thus

$$y=Cos\ s=1-\frac{s^2}{2!}+\frac{s^4}{4!}-\frac{s^6}{6!}+\cdots\tag{3.2}$$

so that

$$0=1-\frac{1}{y}+\frac{s^2}{2!\,y}-\frac{s^4}{4!\,y}+\frac{s^6}{6!\,y}+\cdots\tag{3.3}$$

To be able to suitably factorise (3.3), divide by $(1-1/y)$ to give

$$0=1+\frac{s^2}{2!(y-1)}-\frac{s^4}{4!(y-1)}+\frac{s^6}{6!(y-1)}+\cdots\tag{3.4}$$

Now factorise (3.4) as

$$0=\left(1-\frac{s}{A}\right)\left(1-\frac{s}{B}\right)\left(1-\frac{s}{C}\right)\cdots\tag{3.5}$$

Using Newton's formula, (see Appendix A), to compare terms between (3.4) and (3.5) yields

$$\begin{aligned}&\alpha=0 &&\text{for } s\\&\beta=\frac{1}{2!(y-1)} &&\text{for } s^2\\&\gamma=0 &&\text{for } s^3\\&\delta=-\frac{1}{4!(y-1)} &&\text{for } s^4\end{aligned}\tag{3.6}$$

etc

So that, using Euler's nomenclature, (see Appendix A),

$$
\begin{aligned}
&P = 0 && \text{for } s\\
&Q = -\frac{1}{(y-1)} && \text{for } s^2\\
&R = 0 && \text{for } s^3\\
&S = \frac{1}{2!(y-1)^2} + \frac{1}{3!(y-1)} && \text{for } s^4
\end{aligned}
\qquad (3.7)
$$

The values of *s* which give the same value of *y* in the Cosine function are

$$
\begin{gathered}
A,\ 2\pi-A,\ 2\pi+A,\ 4\pi-A,\ 4\pi+A,\ \ldots \text{ etc}\\
-A,\ -2\pi+A,\ -2\pi-A,\ -4\pi+A,\ -4\pi-A,\ \ldots \text{ etc}
\end{gathered}
\qquad (3.8)
$$

which arranged as a sequence is

$$
\frac{1}{A},-\frac{1}{A},\frac{1}{2\pi-A},-\frac{1}{2\pi-A},\frac{1}{2\pi+A},-\frac{1}{2\pi+A},\frac{1}{4\pi-A},-\frac{1}{4\pi-A},\frac{1}{4\pi+A},-\frac{1}{4\pi A},\ \text{etc} \qquad (3.9)
$$

When this sequence is summed it equals zero which corresponds to $P = 0$, $R = 0$ etc in (3.7). Therefore the only closed forms of infinite series with even exponents can be determined via the Cosine Function. Thus taking terms two at a time, and putting A = π/2 so that $y = \text{Cos}\ \pi/2 = 0$ and therefore in (3.7) $Q = 1$. The summed sequence of (3.9) then becomes

$$
\frac{4}{\pi^2}+\frac{4}{\pi^2}+\frac{4}{3\pi^2}+\frac{4}{3\pi^2}+\cdots+\frac{4}{9\pi^2}+\frac{4}{9\pi^2}+\cdots = Q = 1 \qquad (3.10)
$$

so that

$$
\frac{8}{\pi^2}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\cdots\right) = 1 \qquad (3.11)
$$

and thus

$$
\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\cdots\right) = \frac{\pi^2}{8} \qquad (3.12)
$$

Eq.(3.12) is the sum of the odd terms of ζ(2).

Similarly, taking terms four at a time so that $y = 0$ and therefore $S = 1/3$ leads to

$$
\left(1+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\frac{1}{9^4}+\cdots\right) = \frac{\pi^4}{90} \qquad (3.13)
$$

Appendix B shows the closed forms of the infinite series for four values of *A* each for four values of the exponent *m*.

4.0 Extension to the Tangent Function.

In extending Euler's first method to the Tangent function, the conventional M^{c}Claurin's expansion of the Tangent function cannot be used because it produces an incorrect solution for even values of the exponent, *m*. This is due to the fact that, such an expansion is one of

the ratio of Sine to Cosine, rather than an expansion in which each of these functions are separately represented. To effect this expansion therefore, Euler's version of Tangent must be used.
Thus if y = Sine s and x = Cosine s, then

$$y = s - \frac{s^3}{3!} + \frac{s^5}{5!} - \frac{s^7}{7!} + \cdots \tag{4.1}$$

and

$$x = 1 - \frac{s^2}{2!} + \frac{s^4}{4!} - \frac{s^6}{6!} + \cdots \tag{4.2}$$

From Euler

$$\boldsymbol{tan}\, s = t = \frac{y}{x} \tag{4.3}$$

so that

$$0 = xt - y \tag{4.4}$$

Substitution from (4.1) and (4.2) then gives

$$0 = t - s - \frac{s^2 t}{2!} + \frac{s^3}{3!} + \frac{s^4 t}{4!} - \frac{s^5}{5!} - \frac{s^6 t}{6!} + \cdots \tag{4.5}$$

so that dividing by t results in a form that can be factorised, i.e.

$$0 = 1 - \frac{s}{t} - \frac{s^2}{2!} + \frac{s^3}{3!\,t} + \frac{s^4}{4!} - \frac{s^5}{5!\,t} - \frac{s^6}{6!} + \cdots \tag{4.6}$$

Now factor this as

$$0 = \left(1 - \frac{s}{A}\right)\left(1 - \frac{s}{B}\right)\left(1 - \frac{s}{C}\right)\cdots \tag{4.7}$$

Then comparing terms between (4.6) and (4.7) gives

$$\begin{aligned} \alpha &= \frac{1}{t} \\ \beta &= -\frac{1}{2!} \\ \gamma &= -\frac{1}{3!\,t} \\ \delta &= \frac{1}{4!} \\ &\text{etc} \end{aligned} \tag{4.8}$$

So that from Newton's formula, (See Appendix A),

$$P = \frac{1}{t}$$
$$Q = -\frac{1}{2!}$$
$$R = \frac{1}{t^3} + \frac{1}{t} \qquad (4.9)$$
$$S = \frac{1}{t^4} + \frac{4}{3t^2} + \frac{1}{t}$$

etc

Now the values of s which gives the same value of t in the Tangent function are

$$A,\ \pi + A,\ 2\pi + A,\ 3\pi + A,\ \ldots \qquad (4.10)$$
$$-\pi - A,\ -2\pi - A,\ -3\pi - A,\ \ldots$$

So that as a sequence, (4.10) becomes

$$\frac{1}{A}, \frac{1}{\pi + A}, -\frac{1}{\pi + A}, \frac{1}{2\pi + A}, \frac{1}{-2\pi + A}, \frac{1}{3\pi + A}, \frac{1}{-3\pi + A} \cdots \qquad (4.11)$$

Take terms one at a time with A = π/3 so that $t = \sqrt{3}$, then $P = 1/\sqrt{3}$ and the summed sequence becomes

$$\frac{3}{\pi} + \frac{3}{4\pi} - \frac{3}{2\pi} + \frac{3}{7\pi} - \frac{3}{5\pi} + \frac{3}{10\pi} - \frac{3}{8\pi} + \cdots = \frac{1}{\sqrt{3}} \qquad (4.12)$$

To give

$$1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \frac{1}{10} - \cdots = \frac{\pi}{3\sqrt{3}} \qquad (4.13)$$

Terms taken two at a time give Q = 4/3 and thus

$$1 + \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{8^2} + \frac{1}{10^2} - \cdots = \frac{4\pi^2}{27} \qquad (4.14)$$

Note that in (4.13) and (4.14) the multiples of three have disappeared. Appendix B shows the closed forms of the infinite series obtained for four values of A each for four values of the exponent m.

5.0 Conclusions.

As stated in the introduction, the extension of Euler's first method of solving "The Basel Problem", to the Cosine and Tangent circular functions produces nothing new in this discipline other than the closed forms of a number of new infinite series. While it is the most productive of Euler's three methods, it has the minor disadvantage in that it is not known which infinite series will be summed to a closed form until the numbers are plugged in. Its main attribute is that it can produce the closed form of a virtually unlimited number of related infinite series by varying the values of A and m.

APPENDIX A.

A Brief Summary of Leonhard Euler's First Method of Solving "The Basel Problem".

If $y = Sin\ s$ then via M^cClaurin

$$y = s - \frac{s^3}{3!} + \frac{s^5}{5!} - \frac{s^7}{7!} + \cdots \tag{A.1}$$

so that

$$0 = 1 - \frac{s}{y} + \frac{s^3}{3!\,y} - \frac{s^5}{5!\,y} + \frac{s^7}{7!\,y} + \cdots \tag{A.2}$$

Euler then factored (A.2) as

$$0 = \left(1 - \frac{s}{A}\right)\left(1 - \frac{s}{B}\right)\left(1 - \frac{s}{C}\right)\cdots \tag{A.3}$$

Then comparing terms between (A.2) and (A.3)

$$\alpha = \frac{1}{y} = \frac{1}{A} + \frac{1}{B} + \frac{1}{C} + \cdots \quad \text{Terms one at a time}$$

$$\beta = 0 \quad \text{Terms two at a time} \tag{A.4}$$

$$\gamma = -\frac{1}{3!\,y} \quad \text{Terms three at a time}$$

$$\delta = 0 \quad \text{Terms four at a time}$$

etc

Then using Newton's formula, (see below), and his own nomenclature, Euler produced

$$P = \alpha = \frac{1}{y} \quad \text{Sum of Terms}$$

$$Q = P\alpha - 2\beta = \frac{1}{y^2} \quad \text{Sum of Squares}$$

$$R = Q\alpha - P\beta + 3\gamma = \frac{1}{y^3} - \frac{1}{2y} \quad \text{Sum of Cubes} \tag{A.5}$$

$$S = R\alpha - Q\beta + p\gamma - 4\delta = \frac{1}{y^4} - \frac{2}{3y^2} \quad \text{Sum of Fourths}$$

The values of *s* that produce the same values of *y* are shown in Fig.A.1.

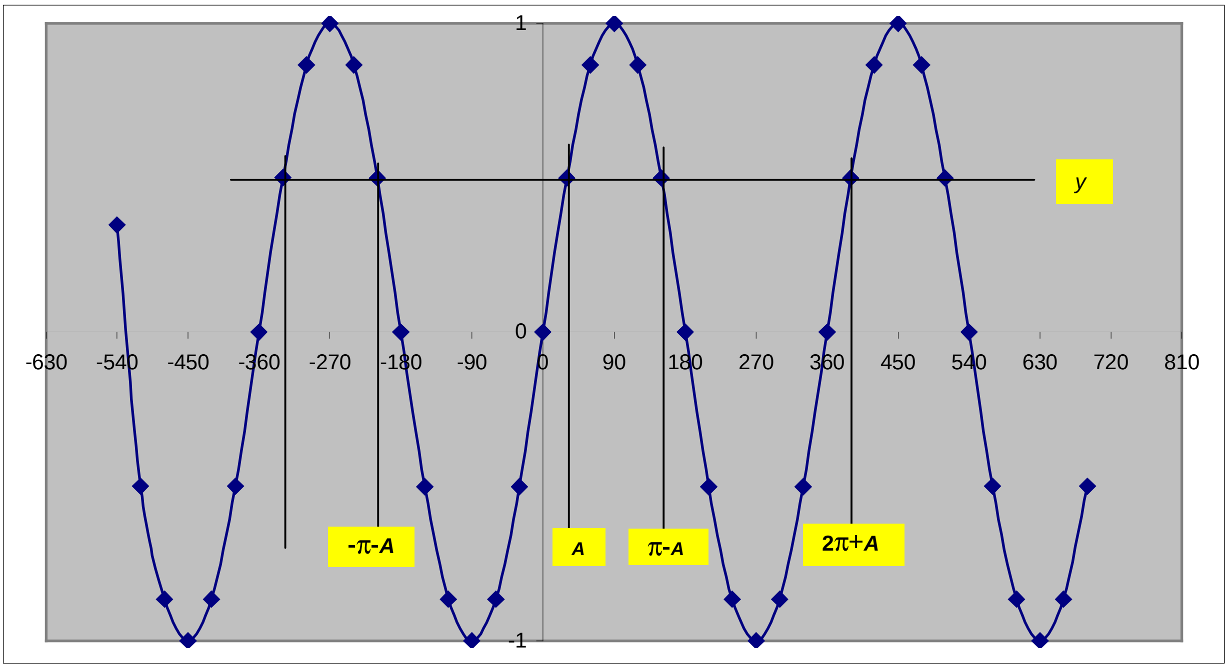

Fig. A.1 - The Sine Curve Showing Those Values of *s* That Produce the Same Values of *y*.

The values of *s* that produce the same value of *y* are from Fig.A.1

$$A,\ \pi-A,\ 2\pi+A,\ 3\pi-A,\ -\pi-A,\ -2\pi+A,\ -3\pi-A\ \ldots\ \text{etc} \tag{A.6}$$

Writing (A.6) as a sequence

$$\frac{1}{A},\frac{1}{\pi-A},\frac{1}{-\pi-A},\frac{1}{2\pi-A},\frac{1}{-2\pi-A},\frac{1}{3\pi-A},\frac{1}{-3\pi-A},\cdots \tag{A.7}$$

Now put $A = \pi/2$ so that $y = 1$ and with (A.6) expressed as a series, the terms taken one at a time, i.e. the sum of the series is equated to P in (A.5) to give

$$\frac{2}{\pi}+\frac{2}{\pi}-\frac{2}{3\pi}+\frac{2}{5\pi}-\frac{2}{3\pi}+\frac{2}{5\pi}-\frac{2}{7\pi}+\cdots=P=\frac{1}{y}=1 \tag{A.8}$$

Which therefore results in

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\cdots=\frac{\pi}{4} \tag{A.9}$$

When terms are taken two at a time, (sum of squares), there results

$$1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\cdots=\frac{\pi^2}{8} \tag{A.10}$$

All of the closed forms of the series produced in this fashion using the Sine function were derived by Leonhard Euler in the eighteenth century. However, for completeness and for comparison with those produced in this paper using the Cosine and Tangent functions, a comparable listing is also shown in Appendix B.

In the development of this method, Euler used some formula derived by Sir Isaac Newton, summarised as follows.

If $a + b + c + d + \ldots$ is a series with sum α, and if the sum of the terms taken two at a time is β, three at a time γ etc then

$$\begin{aligned} a + b + c + d + \cdots &= \alpha \\ a^2 + b^2 + c^2 + d^2 + \cdots &= \alpha^2 - 2\beta \\ a^3 + b^3 + c^3 + d^3 + \cdots &= \alpha^3 - 3\alpha\beta + 3\gamma \end{aligned} \tag{A.11}$$

etc

and Euler represented this as

$$\begin{aligned} P &= \alpha \\ Q &= P\alpha - 2\beta \\ R &= Q\alpha - P\beta + 3\gamma \\ S &= R\alpha - Q\beta + P\gamma - 4\delta \end{aligned} \tag{A.12}$$

etc

A more comprehensive explanation of Euler's first method is shown in [1].

APPENDIX B.

Brief Tables of Infinite Series and Their Closed Forms as Produced by the Sine, Cosine and Tangent Circular Functions.

B.1 Sine.

Function	A	m	Series	Closed Form
Sine	$\pi/2$	1	$1-1/3+1/5-1/7+1/9\cdots$	$\pi/4$
		2	$1+1/3^2+1/5^2+1/7^2+1/9^2\cdots$	$\pi^2/8$
		3	$1-1/3^3+1/5^3-1/7^3+1/9^3\cdots$	$\pi^3/32$
		4	$1+1/3^4+1/5^4+1/7^4+1/9^4\cdots$	$\pi^4/96$
	$\pi/3$	1	$1+1/2-1/4-1/5+1/7+1/8-1/10\cdots$	$2\pi/3\sqrt{3}$
		2	$1+1/2^2+1/4^2+1/5^2+1/7^2+1/8^2+1/10^2\cdots$	$4\pi^2/27$
		3	$1+1/2^3-1/4^3-1/5^3+1/7^3+1/8^3-1/10^3\cdots$	$5\pi^3/81\sqrt{3}$
		4	$1+1/2^4+1/4^4+1/5^4+1/7^4+1/8^4+1/10^4\cdots$	$8\pi^4/729$
	$\pi/4$	1	$1+1/3-1/5-1/7+1/9\cdots$	$\pi/2\sqrt{2}$
		2	$1+1/3^2+1/5^2+1/7^2+1/9^2\cdots$	$\pi^2/8$
		3	$1+1/3^3-1/5^3-1/7^3+1/9^2\cdots$	$\pi^3/32\sqrt{2}$
		4	$1+1/3^4+1/5^4+1/7^4+1/9^4\cdots$	$\pi^4/96$
	$\pi/6$	1	$1+1/5-1/7-1/11+1/13+1/17\cdots$	$\pi/3$
		2	$1+1/5^2+1/7^2+1/11^2+1/13^2+1/17^2\cdots$	$\pi^2/9$
		3	$1+1/5^3-1/7^3-1/11^3+1/13^3+1/17^3\cdots$	$7\pi^3/216$
		4	$1+1/5^4+1/7^4+1/11^4+1/13^4+1/17^4\cdots$	$5\pi^4/486$

B.2 Cosine.

Function	A	m	Series	Closed Form
Cosine	$\pi/2$	1	Series sums to zero for odd ***m***	
		2	$1+1/3^2+1/5^2+1/7^2+1/9^2\cdots$	$\pi^2/8$
		3		
		4	$1+1/3^4+1/5^4+1/7^4+1/9^4\cdots$	$\pi^4/96$
	$\pi/3$	1		
		2	$1+1/5^2+1/7^2+1/11^2+1/13^2+1/17^2\cdots$	$\pi^2/9$
		3		
		4	$1+1/5^4+1/7^4+1/11^4+1/13^4+1/17^4\cdots$	$5\pi^4/486$
	$\pi/4$	1		
		2	$1+1/7^2+1/9^2+1/15^2\cdots$	$\sqrt{2}\,\pi^2/32(\sqrt{2}-1)$
		3		
		4	$1+1/7^4+1/9^4+1/15^4\cdots$	$(4+\sqrt{2})\pi^4/3072(\sqrt{2}-1)^2$
	$\pi/6$	1		
		2	$1+1/11^2+1/13^2+1/23^2\cdots$	$\pi^2/36(2-\sqrt{3})$
		3		
		4	$1+1/11^4+1/13^4+1/23^4\cdots$	$(4+\sqrt{3})\pi^4/7776(2-\sqrt{3})^2$

B.3 Tangent.

Function	A	m	Series	Closed Form
Tangent	$\pi/2$		Tangent $\to$ Infinity	
	$\pi/3$	1	$1-1/2+1/4-1/5+1/7-1/8+1/10\cdots$	$\pi/3\sqrt{3}$
		2	$1+1/2^2+1/4^2+1/5^2+1/7^2+1/8^2+1/10^2\cdots$	$4\pi^2/27$
		3	$1-1/2^3+1/4^3-1/5^3+1/7^3-1/8^3+1/10^3\cdots$	$4\pi^3/81\sqrt{3}$
		4	$1+1/2^4+1/4^4+1/5^4+1/7^4+1/8^4+1/10^4\cdots$	$8\pi^4/729$
	$\pi/4$	1	$1-1/3+1/5-1/7+1/9\cdots$	$\pi/4$
		2	$1+1/3^2+1/5^2+1/7^2+1/9^2\cdots$	$\pi^2/8$
		3	$1-1/3^3+1/5^3-1/7^3+1/9^3\cdots$	$\pi^3/32$
		4	$1+1/3^4+1/5^4+1/7^4+1/9^4\cdots$	$\pi^4/96$
	$\pi/6$	1	$1-1/5+1/7-1/11+1/13-1/17\cdots$	$\pi/2\sqrt{3}$
		2	$1+1/5^2+1/7^2+1/11^2+1/13^2+1/17^2\cdots$	$\pi^2/9$
		3	$1-1/5^3+1/7^3-1/11^3+1/13^3-1/17^3\cdots$	$\pi^3/54\sqrt{3}$
		4	$1+1/5^4+1/7^4+1/11^4+1/13^4+1/17^4\cdots$	$5\pi^4/486$

Within the 36 series in the above three tables, there are a number of duplications. They are listed below

(i) For m = 1 and 3.

Sin(π/2) ≡ Tan(π/4).

(ii) For m = 2 and 4.

Sin(π/2) ≡ Sin(π/4) ≡ Cos(π/2) ≡ Tan(π/4).
Sin(π/3) ≡ Tan(π/3).
Sin(π/6) ≡ Cos(π/3) ≡ Tan(π/6).

Thus there are 14 duplications, leaving 22 unique of which 14 are via Euler's original use of Sine, 4 are via the use of Cosine and 4 are via the use of Tangent.

REFERENCES.

[1] C.Edward Sandifer, *The Early Mathematics of Leonhard Euler*, The Mathematical Association of America, 2007.