Further Deliberations on the Convergence

of the Harmonic Series

and

Commentary on the Nature of the

Euler – Mascheroni Constant.

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Abstract.

Following on from [1], this paper provides a further deliberation on the convergence of the Harmonic series, and also shows the true nature of the Euler – Mascheroni constant.

Contents.

1.0 Introduction.

The determination of the closed form of $\zeta(1)$ in [1], resulted in a value of ~ 419.This was designated as an "of the order of" value because the result of the semi-analytic/empirical method used exhibited a significant level of variability around this value. Therefore, it is necessary to provide an alternative deliberation to provide some level of assurance. The method used here is also a semi-analytic/empirical one very similar to that used in [1], but with the significant difference in that it employs the Euler - Mascheroni constant as one of the participant parameters. The method also involves the introduction of a second constant representing the ratio of $\zeta(1)$ to the area under its curve. It is only these two parameters that are necessary to establish the closed form value of $\zeta(1)$.

This process also reveals that the Euler – Mascheroni constant is the limiting value of a function representing the difference between $\zeta(1)$ and the area under its curve. This function is a very slowly varying function of the term number. The second constant introduced is also a very slowly varying function of the term number. These two functions, "Euler – Mascheroni functions", also enable the determination of the summation value to any desired term number.

2.0 Nomenclature.

The nomenclature used in this paper is as follows.

$\zeta(n)$	Represents any infinite series of the Zeta type.
n	Represents the exponent applied to all terms of the series.
m	Represents any term number of the series.
$k_1(1)$	Represents the Euler – Mascheroni constant.
$f_1(m)$	Represents a Euler – Mascheroni function of which $k_1(1)$ is the limiting value as $m \to \infty$.
$k_2(1)$	Represents the limiting value of a function relating the ratio under the curve to the closed form value of $\zeta(1)$.
$f_2(m)$	Represents the function of which $k_2(1)$ is the limiting value as $m \to \infty$.
$A(n)$	Represents the area under the curve.

3.0 Initial Discussions.

3.1 The Ln(m) Anomaly.

First it is necessary to discuss the problem of the presence of Ln(m) as associated with $\zeta(1)$. Ln(m) is the area under the $\zeta(1)$ curve from term number values of 1 to $1/m$. However, as shown in [1], when $m \to \infty$, it gives a false value of infinity for the area when clearly the area is bounded. To address this problem, consider the $\zeta(1)$ curve as follows.

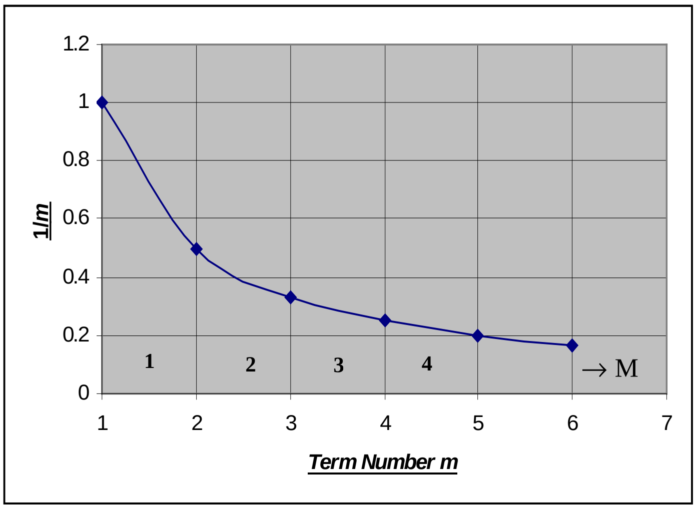

Fig 3.1 – The ζ(1) Simulated Curve.

The total area under the curve is

$$A(1) = \int_1^\infty \frac{1}{m}\,dm = Ln(m)\big|_{m\to\infty} \tag{3.1}$$

Now consider the area as the summation of the elements 1 to M as in Fig. 3.1. The area of Element 1 is

$$A_1(1) = Ln(2) - Ln(1) = Ln(2) \tag{3.2}$$

and of Element 2

$$A_2(1) = Ln(3) - Ln(2) = Ln\left(3/2\right) \\ etc \tag{3.3}$$

so that for some Element M

$$A_M(1) = Ln(m+1) - Ln(m) = Ln\left(1 + \frac{1}{m}\right) \tag{3.4}$$

So that the total area under the curve is

$$A(1) = \sum_1^\infty Ln\left(1 + \frac{1}{m}\right) \tag{3.5}$$

Expressing the logarithmic term in (3.5) in a Newton expansion

$$Ln\left(1 + \frac{1}{m}\right) = \frac{1}{m} - \frac{1}{2m^2} + \frac{1}{3m^3} - \cdots + \frac{1}{pm^p} - \cdots \tag{3.6}$$

So that

$$A(1)=\sum_{1}^{\infty}\left(\frac{1}{m}-\frac{1}{2m^2}+\frac{1}{3m^3}-\cdots+\frac{1}{pm^p}-\cdots\right) \tag{3.7}$$

Now when $m \to \infty$, $A_M \to 0$, so that $A(1)$ is made up of the sum of all other series where m is finite. As all such series are alternating, they will all have a finite sum. Therefore $A(1)$ is finite.
The value of $A(1)$ is given by Euler's modified equation, {(3.14) below}, i.e.

$$A(1)=\zeta(1)-k_1(1) \tag{3.8}$$

and with ζ(1) determined to lie between approximately 419 as determined in [1], and 425 as below in this paper then

$$A(1)\approx 419-k_1(1) \quad to \quad 425-k_1(1) \tag{3.9}$$

Consequently this means that

$$Ln(m)\big|_{m\to\infty}\approx 422-k_1(1) \tag{3.10}$$

3.2 Setting Up the Base Data.

In 1735 Leonard Euler extended his previous work on this subject to derive the following relationship for the approximate sum of the Harmonic series.

$$\zeta(1)=Const+Ln(x)+\frac{1}{2x}-\frac{1}{12x^2}+\frac{1}{120x^4}-\frac{1}{252x^6}+etc \tag{3.11}$$

From this Euler calculated the Euler – Mascheroni constant as, (using x = 10).

$$k_1(1) = 0.5772156649 \text{ etc} \tag{3.12}$$

He then went on to say that when $x \to \infty$

$$\zeta(1)=\ln(\infty)+k_1(1) \tag{3.13}$$

While such a statement is not acceptable today, it can be changed to read

$$\zeta(1)=A(1)+k_1(1) \tag{3.14}$$

Accordingly, a new constant is now introduced as

$$\frac{\zeta(1)}{A(1)}=k_2(1) \tag{3.15}$$

and substitution of (3.15) into (3.14) leads to

$$\zeta(1)=\frac{k_2(1)k_1(1)}{k_2(1)-1} \tag{3.16}$$

So that the closed form of ζ(1) can now be effected by the determination of $k_2(1)$. Determination of $k_2(1)$ is effected in a similar manner to that adopted in [1] by expressing it as

$$k_2(1) = \frac{\pi}{k_{2D}(1)} \tag{3.17}$$

along with similar relationships for ζ(*n*) where *n* is zero and 2 to 8. To obtain these latter parameters note that for $n = 2$ to 8 the area under the curve is given by

$$A(n) = \int_1^\infty \frac{1}{m^n}\,dm = \frac{1}{n-1} \tag{3.18}$$

All this results in the following table for $k_{2D}(n)$ where $k_{2D}(n)$ is expressed as

$$k_{2D}(n) = \frac{\pi^n A(n)}{\zeta(n)} \tag{3.19}$$

n	$\boldsymbol{k_{2D}(n)}$
0	0
1	$k_{2D}(1)$
2	6
3	12.8971756
4	30
5	73.128125
6	189
7	499.254745
8	1350

Table 3.1 - $k_{2D}(n)$ vs *n*

The above numerical values in the table can be graphed and accurately curve fitted to produce an algorithm, from which $k_{2D}(1)$ can be obtained. However, as in [1], the sensitivity of the curve at $n = 1$ is such that just graphing the numerical values in Table 3.1 is too coarse to provide a reliable algorithm. Consequently, the curve is augmented by a series of trial values of $k_{2D}(1)$ from which the most suitable value can be identified via minimisation of overall errors.

After summing 100,000,000 terms of the Harmonic series, the sum is just under 19, and so it is clear that $k_2(1)$ must be > 1.03133167. Also as $k_2(1)$ must be ≥ 0, $k_{2D}(1)$ must be ≤ π, i.e.

$$3.046151636 < k_{2D}(1) < \pi \tag{3.20}$$

$k_{2D}(1)$ therefore lies in a very narrow band. The data points to be graphed will straddle this range to provided the best visibility of the trend. Initially therefore, the trial $k_{2D}(1)$ data points will be.

1, 1.5, 2, 2.5, 3, and 3.5

These will set a standard before the above expected range is explored in more detail, to find the value which gives the minimum overall errors. The values chosen within the expected range were largely at random but exploring the full range.

4.0 Determination of ζ(1).

The curve fit results for the initial set of trial values of $k_{2D}(1)$ are shown in Appendix A, Table A.1.These show that the algorithms for $k_{2D}(n)$ are all 8 order polynomials and produce errors of between 1E-12 to 1E –10 with the exception of $k_{2D}(3)$ of just over 1E-8.

These were adequate to allow the expected range of $k_{2D}(1)$ to be explored in detail. The results of this is shown in Appendix A, Table A.2. Here the algoritms are again 8 order polynomials with errors for $k_{2D}(n)$ ranging from 1E-11 to zero.

5.0 Discussion of Results.

5.1 The Closed Form Value of ζ(1).

To analyse the above results the error values for all $k_{2D}(1)$ trials were consolidated in Appendix B, Table B.1. These clearly show that the minimum errors mostly lie in the expected range of $k_{2D}(1)$. However, albeit this being so there is no clear indication of which is the most suitable value. To progress, Table B.2(a) was constructed which put all the error values in numerical order from smallest to largest for all eight exponents. From this table another was constructed in which a points score, (of from 1 to 11), was allocated for each of the eight exponents according to their error level for each $k_{2D}(1)$ trial value. These were then averaged. The final result showed that there were two $k_{2D}(1)$ trial values within the expected range that had the lowest points score. Each had three lowest error values and one error level of zero. The former corresponded to a closed form value for ζ(1) of approximately 425 while the latter to a value of approximately 550. While the results for these two trial values are very close, the former is slightly better and also its closed form value of ζ(1) corresponds very well with the value determined in [1]. Accordingly, the former value was considered slightly the superior of the two. The final points score results are shown in Fig. 5.1 below.

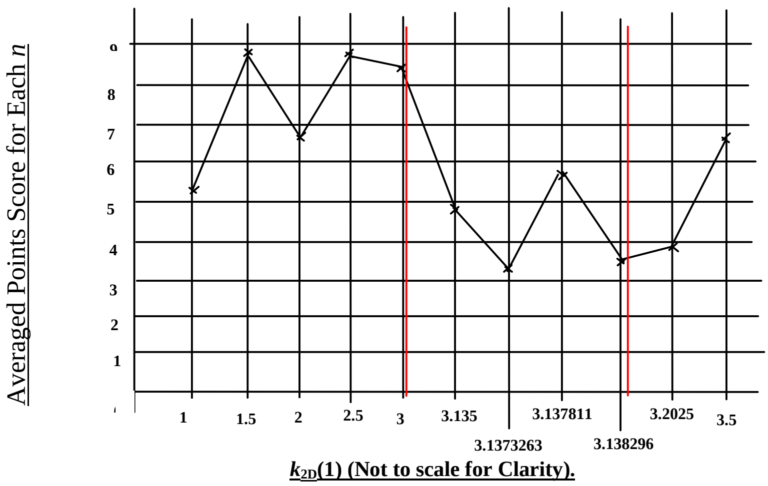

Fig. 5.1 - ζ(1) $k_{2D}(1)$ Points Score vs $k_{2D}(1)$ Trial Values.
(The expected range lies between the two red lines.)

5.2 The Euler - Mascheroni Constant.

There is one other feature of note that arises from this dissertation. It concerns the Euler – Mascheroni constant. From the analysis it became clear that this constant is actually the limiting value, (as m $\rightarrow \infty$), of a very slowly varying function of the term number *m*. When m = 2 the value of this function is 0.806852819 so that the variation over the entire range of *m* is only -0.22963715. This means that this function, $f_1(m)$, could be approximated with a very simple linear relationship of the form.

$$f_1(m) \approx a + \frac{b}{m} \tag{5.1}$$

Similarly, the second constant introduced, is also seen to be a slowly varying function of the term number, and could also be approximated with

$$f_2(m) \approx a + \frac{b}{m^q} \tag{5.2}$$

These features are discussed further in Appendix C.

6.0 Conclusions.

In dispelling the anomaly concerning the presence of Ln(*m*) in its relationship with $\zeta(1)$, together with the discussions on the convergence/divergence tests in [2] as applied to the Harmonic series, it is proposed that the convergence of $\zeta(1)$ to a finite closed form has been positively established. The value of the closed form is however, less certain.

While the result here is in very close agreement with that in [1], once again it exhibits a high degree of variability across the expected range of $k_{2D}(1)$. As in [1], this is due to the extreme sensitivity to small fractional changes in the term number around the exponent of unity. Accordingly, even with the excellent correspondence of these results, it cannot be claimed that the true value has been established, and this will probably only become known when a fully analytic solution has been devised.

With regard to the "Euler – Mascheroni functions", the linear relationships proposed in Section 5.1 are of course only approximations, as it is not possible to derive a function of this kind covering the whole range of from m = 1 to ∞. Nevertheless, once the function approximation has been established for some range of term numbers, the partial sum to any value of *m* within that range can be calculated. Note that this concept applies to any and all convergent infinite series. It may also apply to divergent series, but this has not been explored.

Appendix A.

Results of the Curve Fit Algorithms for the Initial Trial Values of $k_{2D}(1)$ Together with Error Results.

name:	Polynomial Regression (degree=8)		$k_{2d}(1) = 1$			
equation:	a + b*x + c*x^2 + ...					
Parameters:			$k_{2D}(n)$.	n	Calculated $k_{2D}(n)$.	Error
a =	-5.2295945352E-12		0	0	-1.421085471520200E-12	1.4210854715E-12
b =	-2.1390881905E+01		1	1	1.000000000002950E+00	2.9500846210E-12
c =	5.0360678841E+01		6	2	5.999999999998290E+00	1.7097434579E-12
d =	-4.3325854083E+01		12.89717564	3	1.289717566000150E+01	2.0001499124E-08
e =	1.9804276611E+01		30	4	2.999999999999880E+01	1.2008172234E-12
f =	-5.1696361458E+00		73.128125	5	7.312812500000340E+01	3.3963942769E-12
g =	7.8317138195E-01		189	6	1.890000000000060E+02	5.9969806898E-12
h =	-6.4000366072E-02		499.254745	7	4.992547450000350E+02	3.4958702599E-11
i =	2.2456656746E-03		1350	8	1.350000000000040E+03	4.0017766878E-11
Standard Error:	0.0000000000E+00					
Correlation Coefficient (r):	1.0000000000E+00					

name:	Polynomial Regression (degree=8)		$k_{2d}(1) = 1.5$			
equation:	a + b*x + c*x^2 + ...					
Parameters:			$k_{2D}(n)$.	n	Calculated $k_{2D}(n)$.	Error
a =	-4.3769432523E-12		0	0	-4.604316927725440E-12	4.6043169277E-12
b =	-1.3390881905E+01		1.5	1	1.499999999987260E+00	1.2740031252E-11
c =	3.6617821698E+01		6	2	5.999999999995420E+00	4.5803361104E-12
d =	-3.3631409639E+01		12.89717564	3	1.289717566000010E+01	2.0000099354E-08
e =	1.6148721056E+01		30	4	2.999999999999640E+01	3.5988989566E-12
f =	-4.3710250347E+00		73.128125	5	7.312812499999450E+01	5.4996007748E-12
g =	6.8178249306E-01		189	6	1.890000000000110E+02	1.0999201550E-11
h =	-5.7055921627E-02		499.254745	7	4.992547450000330E+02	3.2969182939E-11
i =	2.0472529762E-03		1350	8	1.350000000000060E+03	6.0026650317E-11
Standard Error:	0.0000000000E+00					
Correlation Coefficient (r):	1.0000000000E+00					

Table A.1 - Curve Fit Algorithms for the Initial Trial Values of $k_{2D}(1)$ Together with Error Results.

name:	Polynomial Regression (degree=8)	
equation:	a + b*x + c*x^2 + ...	
Parameters:		
a =	-3.1263880373E-13	
b =	-5.3908819047E+00	
c =	2.2874964555E+01	
d =	-2.3936965194E+01	
e =	1.2493165500E+01	
f =	-3.5724139236E+00	
g =	5.8039360416E-01	
h =	-5.0111477182E-02	
i =	1.8488402778E-03	
Standard Error:	0.0000000000E+00	
Correlation Coefficient (r):	1.0000000000E+00	

$k_{2d}(1) = 2$			
$k_{2D}(n)$.	n	Calculated $k_{2D}(n)$.	Error
0	0	2.046363078989080E-12	2.0463630790E-12
2	1	1.999999999992840E+00	7.1600503304E-12
6	2	6.000000000002010E+00	2.0099477638E-12
12.89717564	3	1.289717566000300E+01	2.0003000145E-08
30	4	2.999999999999870E+01	1.3002932064E-12
73.128125	5	7.312812500000280E+01	2.7995383789E-12
189	6	1.890000000000080E+02	7.9865003499E-12
499.254745	7	4.992547450000390E+02	3.8994585339E-11
1350	8	1.350000000000050E+03	5.0022208598E-11

name:	Polynomial Regression (degree=8)	
equation:	a + b*x + c*x^2 + ...	
Parameters:		
a =	2.0463630790E-12	
b =	2.6091180952E+00	
c =	9.1321074126E+00	
d =	-1.4242520750E+01	
e =	8.8376099444E+00	
f =	-2.7738028125E+00	
g =	4.7900471528E-01	
h =	-4.3167032738E-02	
i =	1.6504275794E-03	
Standard Error:	0.0000000000E+00	
Correlation Coefficient (r):	1.0000000000E+00	

$k_{2d}(1) = 2.5$			
$k_{2D}(n)$.	n	Calculated $k_{2D}(n)$.	Error
0	0	-3.865352482534940E-12	3.8653524825E-12
2.5	1	2.499999999989340E+00	1.0659917393E-11
6	2	5.999999999996160E+00	3.8395953084E-12
12.89717564	3	1.289717566000060E+01	2.0000600287E-08
30	4	2.999999999999850E+01	1.4992451725E-12
73.128125	5	7.312812499999970E+01	2.9842794902E-13
189	6	1.890000000000230E+02	2.2993162929E-11
499.254745	7	4.992547450000520E+02	5.2011728258E-11
1350	8	1.350000000000110E+03	1.1004885891E-10

Table A.1 - Curve Fit Algorithms for the Initial Trial Values of $k_{2D}(1)$ Together with Error Results.

			$k_{2d}(1) = 3$			
name:	Polynomial Regression (degree=8)					
equation:	a + b*x + c*x^2 + ...					
Parameters:			$k_{2D}(n)$.	n	Calculated $k_{2D}(n)$.	Error
a =	1.9326762413E-12		0	0	-3.410605131648480E-12	3.4106051316E-12
b =	1.0609118095E+01		3	1	2.999999999990240E+00	9.7601926541E-12
c =	-4.6107497302E+00		6	2	5.999999999996460E+00	3.5402791809E-12
d =	-4.5480763055E+00		12.89717564	3	1.289717566000020E+01	2.0000198830E-08
e =	5.1820543889E+00		30	4	2.999999999999750E+01	2.5011104299E-12
f =	-1.9751917014E+00		73.128125	5	7.312812499999800E+01	2.0037305148E-12
g =	3.7761582639E-01		189	6	1.890000000000180E+02	1.7990942069E-11
h =	-3.6222588294E-02		499.254745	7	4.992547450000450E+02	4.4963144319E-11
i =	1.4520148809E-03		1350	8	1.350000000000090E+03	9.0039975476E-11
Standard Error:	0.0000000000E+00					
Correlation Coefficient (r):	1.0000000000E+00					

			$k_{2d}(1) = 3.2025$			
name:	Polynomial Regression (degree=8)					
equation:	a + b*x + c*x^2 + ...					
Parameters:			$k_{2D}(n)$.	n	Calculated $k_{2D}(n)$.	Error
a =	1.93E-12		0	0	-5.1159076975E-13	5.1159076975E-13
b =	-3.77E+00		3.2025	1	3.2025000000E+00	1.1999290450E-12
c =	2.01E+01		6	2	6.0000000000E+00	1.1599610161E-12
d =	-2.20E+01		12.89717564	3	1.2897175640E+01	1.0995648836E-12
e =	1.18E+01		30	4	3.0000000000E+01	6.0040861172E-13
f =	-3.41E+00		73.128125	5	7.3128125000E+01	3.0979663279E-12
g =	5.60E-01		189	6	1.8900000000E+02	6.9917405199E-12
h =	-4.87E-02		499.254745	7	4.9925474500E+02	3.3992364479E-11
i =	1.81E-03		1350	8	1.3500000000E+03	5.0022208598E-11
Standard Error:	0.00E+00					
Correlation Coefficient (r):	1.00E+00					

Table A.1 - Curve Fit Algorithms for the Initial Trial Values of $k_{2D}(1)$ Together with Error Results.

Name	Polynomial Regression (degree=8)	
equation:	a + b*x + c*x^2 + ...	
Parameters:		
a =	1.9326762413E-12	
b =	6.6091180952E+00	
c =	2.2606788412E+00	
d =	-9.3952985277E+00	
e =	7.0098321666E+00	
f =	-2.3744972569E+00	
g =	4.2831027083E-01	
h =	-3.9694810516E-02	
i =	1.5512212302E-03	
Standard Error:	0.0000000000E+00	
Correlation Coefficient (r):	1.0000000000E+00	

$k_{2d}(1) = 3.5$			
$k_{2D}(n)$.	n	Calculated $k_{2D}(n)$.	Error
0	0	-3.012701199622820E-12	3.0127011996E-12
3.5	1	3.499999999991130E+00	8.8697937883E-12
6	2	5.999999999996880E+00	3.1201707884E-12
12.89717564	3	1.289717566000050E+01	2.0000499035E-08
30	4	2.999999999999810E+01	1.9007018182E-12
73.128125	5	7.312812499999760E+01	2.4016344469E-12
189	6	1.890000000000120E+02	1.1993961380E-11
499.254745	7	4.992547450000290E+02	2.8990143619E-11
1350	8	1.350000000000060E+03	6.0026650317E-11

Table A.1 - Curve Fit Algorithms for the Initial Trial Values of $k_{2D}(1)$ Together with Error Results.

Expected Range

name:	Polynomial Regression (degree=8)	
equation:	a + b*x + c*x^2 + ...	
Parameters:		
a =	1.93E-12	
b =	-3.77E+00	
c =	2.01E+01	
d =	-2.20E+01	
e =	1.18E+01	
f =	-3.41E+00	
g =	5.60E-01	
h =	-4.87E-02	
i =	1.81E-03	
Standard Error:	0.00E+00	
Correlation Coefficient (r):	1.00E+00	

$k_{2d}(1)) = 3.135$			
$k_{2D}(n)$.	n	Calculated $k_{2D}(n)$.	Error
0	0	-6.5369931690E-13	6.5369931690E-13
3.135	1	3.1350000000E+00	1.1404210909E-12
6	2	6.0000000000E+00	1.2096990076E-12
12.89717564	3	1.2897175000E+01	6.3999880062E-07
30	4	3.0000000000E+01	2.9842794902E-13
73.128125	5	7.3128125000E+01	2.7000623959E-12
189	6	1.8900000000E+02	9.0096818894E-12
499.254745	7	4.9925474500E+02	3.3992364479E-11
1350	8	1.3500000000E+03	4.0017766878E-11

Table A.2 - Curve Fit Algorithms for the Expected Range of Trial Values of $k_{2D}(1)$ Together with Error Results.

name:	Polynomial Regression (degree=8)	
equation:	a + b*x + c*x^2 + ...	
Parameters:		
a =	-3.13E-13	
b =	-4.29E+00	
c =	2.10E+01	
d =	-2.26E+01	
e =	1.20E+01	
f =	-3.46E+00	
g =	5.66E-01	
h =	-4.92E-02	
i =	1.82E-03	
Standard Error:	0.00E+00	
Correlation Coefficient (r):	1.00E+00	

k_{2d}(1) = 3.1373263			
$k_{2D}(n)$.	n	Calculated $k_{2D}(n)$.	Error
0	0	-3.1263880373E-13	3.1263880373E-13
3.1373263	1	3.1373263000E+00	1.2501111257E-12
6	2	6.0000000000E+00	1.0302869669E-12
12.89717564	3	1.2897175640E+01	1.4992451725E-12
30	4	3.0000000000E+01	0.0000000000E+00
73.128125	5	7.3128125000E+01	2.7000623959E-12
189	6	1.8900000000E+02	7.9865003499E-12
499.254745	7	4.9925474500E+02	3.7971403799E-11
1350	8	1.3500000000E+03	4.0017766878E-11

name:	Polynomial Regression (degree=8)	
equation:	a + b*x + c*x^2 + ...	
Parameters:		
a =	-3.69E-13	
b =	-4.29E+00	
c =	2.10E+01	
d =	-2.26E+01	
e =	1.20E+01	
f =	-3.46E+00	
g =	5.66E-01	
h =	-4.92E-02	
i =	1.82E-03	
Standard Error:	0.00E+00	
Correlation Coefficient (r):	1.00E+00	

k_{2d}(1) = 3.137811			
$k_{2D}(n)$.	n	Calculated $k_{2D}(n)$.	Error
0	0	2.1600499167E-12	2.1600499167E-12
3.137811	1	3.1378110000E+00	6.0902394239E-12
6	2	6.0000000000E+00	1.9797496975E-12
12.89717564	3	1.2897175640E+01	2.5988100560E-12
30	4	3.0000000000E+01	7.9936057773E-13
73.128125	5	7.3128125000E+01	4.0074610297E-12
189	6	1.8900000000E+02	9.0096818894E-12
499.254745	7	4.9925474500E+02	2.7000623959E-11
1350	8	1.3500000000E+03	5.0022208598E-11

Table A.2 - Curve Fit Algorithms for the Expected Range of Trial Values of k_{2D}(1) Together with Error Results.

name:	Polynomial Regression (degree=8)	
equation:	a + b*x + c*x^2 + ...	
Parameters:		
a =	-5.40E-13	
b =	-4.28E+00	
c =	2.10E+01	
d =	-2.26E+01	
e =	1.20E+01	
f =	-3.46E+00	
g =	5.66E-01	
h =	-4.92E-02	
i =	1.82E-03	
Standard Error:	0.00E+00	
Correlation Coefficient (r):	1.00E+00	

$k_{2d}(1)$ = 3.138296			
$k_{2D}(n)$.	n	Calculated $k_{2D}(n)$.	Error
0	0	1.9895196601E-12	1.9895196601E-12
3.138296	1	3.1382960000E+00	6.1297633636E-12
6	2	6.0000000000E+00	1.7301715616E-12
12.89717564	3	1.2897175640E+01	2.3998580900E-12
30	4	3.0000000000E+01	1.3002932064E-12
73.128125	5	7.3128125000E+01	1.0089706848E-12
189	6	1.8900000000E+02	0.0000000000E+00
499.254745	7	4.9925474500E+02	1.3983481040E-11
1350	8	1.3500000000E+03	1.0004441720E-11

Table A.2 - Curve Fit Algorithms for the Expected Range of Trial Values of $k_{2D}(1)$ Together with Error Results.

Appendix B

Details of the Error Table Analysis for All Trial Values of $k_{2D}(1)$.

$k_{2d}(1)$ \ Exp n	0	1	2	3	4
1	1.4210854715E-12	2.9500846210E-12	1.7097434579E-12	2.0001499124E-08	1.2008172234E-12
1.5	**4.6043169277E-12**	**1.2740031252E-11**	**4.5803361104E-12**	2.0000099354E-08	**3.5988989566E-12**
2	2.0463630790E-12	7.1600503304E-12	2.0099477638E-12	2.0003000145E-08	1.3002932064E-12
2.5	3.8653524825E-12	1.0659917393E-11	3.8395953084E-12	2.0000600287E-08	1.4992451725E-12
3	3.4106051316E-12	9.7601926541E-12	3.5402791809E-12	2.0000198830E-08	2.5011104299E-12
3.135	6.5369931690E-13	**1.1404210909E-12**	1.2096990076E-12	**6.3999880062E-07**	2.9842794902E-13
3.1373263	**3.1263880373E-13**	1.2501111257E-12	**1.0302869669E-12**	1.4992451725E-12	**0.0000000000E+00**
3.137811	2.1600499167E-12	6.0902394239E-12	1.9797496975E-12	2.5988100560E-12	7.9936057773E-13
3.138296	1.9895196601E-12	6.1297633636E-12	1.7301715616E-12	2.3998580900E-12	1.3002932064E-12
3.2025	5.1159076975E-13	1.1999290450E-12	1.1599610161E-12	**1.0995648836E-12**	6.0040861172E-13
3.5	3.0127011996E-12	8.8697937883E-12	3.1201707884E-12	2.0000499035E-08	1.9007018182E-12

$k_{2d}(1)$ \ Exp n	5	6	7	8
1	3.3963942769E-12	5.9969806898E-12	3.4958702599E-11	4.0017766878E-11
1.5	**5.4996007748E-12**	1.0999201550E-11	3.2969182939E-11	6.0026650317E-11
2	2.7995383789E-12	7.9865003499E-12	3.8994585339E-11	5.0022208598E-11
2.5	**2.9842794902E-13**	**2.2993162929E-11**	**5.2011728258E-11**	**1.1004885891E-10**
3	2.0037305148E-12	1.7990942069E-11	4.4963144319E-11	9.0039975476E-11
3.135	2.7000623959E-12	9.0096818894E-12	3.3992364479E-11	4.0017766878E-11
3.1373263	2.7000623959E-12	7.9865003499E-12	3.7971403799E-11	4.0017766878E-11
3.137811	4.0074610297E-12	9.0096818894E-12	2.7000623959E-11	5.0022208598E-11
3.138296	1.0089706848E-12	**0.0000000000E+00**	**1.3983481040E-11**	**1.0004441720E-11**
3.2025	3.0979663279E-12	6.9917405199E-12	3.3992364479E-11	5.0022208598E-11
3.5	2.4016344469E-12	1.1993961380E-11	2.8990143619E-11	6.0026650317E-11

Table B.1 - Error Table for All Trial Values of $k_{2D}(1)$.

(Note: The figures in red are the minimum errors, blue are the maximum errors).

Trial k_{2D}(1)) Denominator Values	Exponent 0	Trial k_{2D}(1) Denominator Values	Exponent 1	Trial k_{2D}(1) Denominator Values	Exponent 2	Trial k_{2D}(1) Denominator Values	Exponent 3
3.1373263	3.1263880373E-13	**3.135**	1.1404210909E-12	**3.1373263**	1.0302869669E-12	**3.2025**	1.0995648836E-12
3.2025	5.1159076975E-13	**3.2025**	1.1999290450E-12	**3.2025**	1.1599610161E-12	**3.137326**	1.4992451725E-12
3.135	6.5369931690E-13	**3.1373263**	1.2501111257E-12	**3.135**	1.2096990076E-12	**3.138296**	2.3998580900E-12
1	1.4210854715E-12	**1**	2.9500846210E-12	**1**	1.7097434579E-12	**3.137811**	2.5988100560E-12
3.138296	1.9895196601E-12	**3.137811**	6.0902394239E-12	**3.138296**	1.7301715616E-12	**1.5**	2.0000099354E-08
2	2.0463630790E-12	**3.138296**	6.1297633636E-12	**3.137811**	1.9797496975E-12	**3**	2.0000198830E-08
3.137811	2.1600499167E-12	**2**	7.1600503304E-12	**2**	2.0099477638E-12	**3.5**	2.0000499035E-08
3.5	3.0127011996E-12	**3.5**	8.8697937883E-12	**3.5**	3.1201707884E-12	**2.5**	2.0000600287E-08
3	3.4106051316E-12	**3**	9.7601926541E-12	**3**	3.5402791809E-12	**1**	2.0001499124E-08
2.5	3.8653524825E-12	**2.5**	1.0659917393E-11	**2.5**	3.8395953084E-12	**2**	2.0003000145E-08
1.5	4.6043169277E-12	**1.5**	1.2740031252E-11	**1.5**	4.5803361104E-12	**3.135**	6.3999880062E-07

Trial k_{2D}(1) Denominator Values	Exponent 4	Trial k_{2D}(1) Denominator Values	Exponent 5	Trial k_{2D}(1) Denominator Values	Exponent 6	Trial k_{2D}(1) Denominator Values	Exponent 7	Trial k_{2D}(1) Denominator Values	Exponent 8
3.1373263	0.0000000000E+00	**2.5**	2.9842794902E-13	**3.138296**	0.0000000000E+00	**3.138296**	1.3983481040E-11	**3.138296**	1.0004441720E-11
3.135	2.9842794902E-13	**3.138296**	1.0089706848E-12	**1**	5.9969806898E-12	**3.137811**	2.7000623959E-11	**1**	4.0017766878E-11
3.2025	6.0040861172E-13	**3**	2.0037305148E-12	**3.2025**	6.9917405199E-12	**3.5**	2.8990143619E-11	**3.135**	4.0017766878E-11
3.137811	7.9936057773E-13	**3.5**	2.4016344469E-12	**2**	7.9865003499E-12	**1.5**	3.2969182939E-11	**3.1373263**	4.0017766878E-11
1	1.2008172234E-12	**3.1373263**	2.7000623959E-12	**3.1373263**	7.9865003499E-12	**3.135**	3.3992364479E-11	**2**	5.0022208598E-11
2	1.3002932064E-12	**3.135**	2.7000623959E-12	**3.135**	9.0096818894E-12	**3.2025**	3.3992364479E-11	**3.2025**	5.0022208598E-11
3.138296	1.3002932064E-12	**2**	2.7995383789E-12	**3.137811**	9.0096818894E-12	**1**	3.4958702599E-11	**3.137811**	5.0022208598E-11
2.5	1.4992451725E-12	**3.2025**	3.0979663279E-12	**1.5**	1.0999201550E-11	**3.1373263**	3.7971403799E-11	**1.5**	6.0026650317E-11
3.5	1.9007018182E-12	**1**	3.3963942769E-12	**3.5**	1.1993961380E-11	**2**	3.8994585339E-11	**3.5**	6.0026650317E-11
3	2.5011104299E-12	**3.137811**	4.0074610297E-12	**3**	1.7990942069E-11	**3**	4.4963144319E-11	**3**	9.0039975476E-11
1.5	3.5988989566E-12	**1.5**	5.4996007748E-12	**2.5**	2.2993162929E-11	**2.5**	5.2011728258E-11	**2.5**	1.1004885891E-10

Table B.2(a) - k_{2D}(1).Error Values Ordered from Smallest to Largest.

Exp't / $k_{2D}(1)$	0	1	2	3	4	5	6	7	8	Average
1	4	4	4	9	5	9	2	7	2	**5.11**
1.5	11	11	11	5	11	11	8	4	8	**8.89**
2	6	7	7	10	6	7	4	9	5	**6.78**
2.5	10	10	10	8	8	1	11	11	11	**8.89**
3	9	9	9	6	10	3	10	10	10	**8.44**
3.135	3	1	3	11	2	6	6	5	3	**4.44**
3.1373263	**1**	**3**	**1**	**2**	**1**	**5**	**5**	**8**	**4**	**3.33**
3.137811	7	5	6	4	4	10	7	2	7	**5.78**
3.138296	5	6	5	3	7	2	1	1	1	**3.44**
3.2025	2	2	2	1	3	8	3	6	6	**3.67**
3.5	8	8	8	7	9	4	9	3	9	**7.22**

Table B.2(b) – Points Score for all Values of $k_{2d}(1)$ for all Exponents.

Appendix C.

The Constants $k_1(n)$ and $k_2(n)$, (n = 2 to 8).

Because the closed forms of $\zeta(n)$ are known and the area under their curves are easily computed, the values of $k_1(n)$ and $k_2(n)$ for them are readily calculated and are shown in the following table.

n	**Closed Form**	**Area Under the Curve**	$k_1(n)$	$k_2(n)$
2	1.644934067	1	0.6449341	1.6449341
3	1.202056851	1/2	0.7020568	2.4041137
4	1.082323234	1/3	0.7489899	3.2469697
5	1.029583496	1/4	0.7795834	4.1183339
6	1.017343062	1/5	0.8173431	5.0867153
7	1.008349277	1/6	0.8416826	6.0500957
8	1.004077356	1/7	0.8612202	7.0285415

Table C.1 – $k_1(n)$ and $k_2(n)$ for $\zeta(n)$ for Exponents n from 2 to 8.

For all other $\zeta(n)$ series the closed form value can be obtained from [4], and the area under the curve is given by (3.18), thus allowing $k_1(n)$ and $k_2(n)$ to be determined for any $\zeta(n)$ up to n = 30, (the maximum in [4]). From this information it may be possible to construct a relationship between n and $k_1(n)$ and $k_2(n)$ so permitting the closed form for all $\zeta(n)$ for n up to ∞ to be approximated. However, there will always be some degree of error arising when using such constructs.

References.

[1] P.G.Bass, *Determination of an Approximate Value for the Closed Form of the Harmonic Series, (Assuming Convergence),* www.relativitydomains.com.

[2] C.Edward Sandifer, *The Early Mathematics of Leonard Euler*, The Mathematics Association of America, 2007.

[3] P.G.Bass, *Determination of the Limiting Divergent Infinite Series, and a Review of the Divergency of the Harmonic Series,* www.relativitydomains.com.

[4] P.G.Bass, *Closed Forms of the Zeta and Eta Infinite Series Via a Modified Gamma Function Relationship,* www.realtivitydomains.com.