PSEUDO CLOSED FORMS FOR THE

ETA AND ZETA INFINITE SERIES

FOR ALL VALUES OF THE EXPONENTS

BOTH ODD AND EVEN.

Peter G.Bass

www.relativitydomains.com

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ABSTRACT.

This paper develops pseudo closed forms for both the Eta and Zeta infinite series applicable to all values of the exponents, both odd and even. The method used is recursive integration, together with a formula developed by Leonhard Euler for the evaluation of infinite series.

CONTENTS.

1.0 Introduction.

Subsequent to Leonhard Euler's 1734 solution of the Basel Problem, the closed form of Zeta(2), (and all other even values of the exponent), many mathematicians since have investigated solutions for the Zeta Function for all odd values of the exponent. This however, has remained elusive up to the present time, and has now only been partly solved using an analytical process different from those used by Euler and later investigators.
The method used here is in two parts. The first is that shown in [1], recursive integration, which has been modified slightly to avoid the difficulty posed by polylogarithmic integrals. The second part is the application of a formula developed by Leonhard Euler, to determine the closed form of any series from a consideration of its general term. Because the series being investigated here are infinite, this latter application only provides a pseudo closed form in that part of the result is numerical.

2.0 Preamble and Definitions.

Euler's methods, and those of many later investigators, of solving the Basel Problem et al, all used circular functions so resulting in solutions in terms of the parameter π. It was shown in [2] that the exponents for which these methods were applicable conformed to the following relationship

$$m = pB^q \tag{2.1}$$

where

- m is the exponent of the series.
- p is a prime number.
- B is the Base Number, (defined in [2]).
- q is any natural number.

As it is clear that an odd exponent cannot be represented by (2.1), use of methods employing circular functions cannot provide solutions for the closed forms of these series with odd exponents.
The only current method therefore applicable, is that of [1], recursive integration. However, there are two minor difficulties associated with its use. The first is that it must start from a base series for which the closed form is already known. For the Zeta function, the base series, $\zeta(1)$, is divergent and cannot therefore be used. Consequently, the base series to be used here is that of the Eta function, $\eta(1)$. Conversion to results for the Zeta Function can then be made using the usual relationship. The use of $\eta(1)$ raises the second difficulty in that the first integration required is that of a polylogarithmic integral, which was shown in [3] to have no known solution in terms of elementary functions. Hence the recursive integration process used here has been modified to avoid this difficulty.

To simply analysis the following definitions are made. In Sir Isaac Newton's expansion of $Ln(1 + x)$, put

$$Ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots = x\eta_1(x) \tag{2.2}$$

So that

$$\eta_1(x) = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \frac{x^4}{5} - \ldots \tag{2.3}$$

and then

$$\int \eta_1(x)\,dx = x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \frac{x^5}{5^2} - \ldots = x\eta_2(x) \tag{2.4}$$

Consequently, it is clear that

$$\int \eta_m(x)\,dx = x\eta_{(m+1)}(x) \tag{2.5}$$

3.0 Development of Selected Infinite Series Closed Forms.

3.1 The Logarithmic Version of the Closed Form of Eta(2).

The closed form of $\eta(2)$ is already well known as $\pi^2/_{12}$. It is however, developed here because, (i) part of the analysis is required later to progress to higher exponents, (ii) it produces the logarithmic version of the closed form, and (iii) it is instrumental in clearly demonstrating the application of this method.

From (2.2)

$$x\eta_1(x) = Ln(1+x) \tag{3.1}$$

Differentiate

$$\eta_1(x) + x\eta_1'(x) = \frac{1}{1+x} \tag{3.2}$$

where

$$\eta_1'(x) = \frac{d\eta_1(x)}{dx} \tag{3.3}$$

The left hand side of (3.2) is an exact differential which is reconfigured simply thus

$$\eta_1(x) = \frac{1}{1+x} - x\eta_1'(x) \tag{3.4}$$

Now re-integrate (3.4) to give

$$x\eta_2(x) = Ln(1+x) - \int x\eta_1'(x)\,dx + k \tag{3.5}$$

From (2.3)

$$\eta_1'(x) = -\frac{1}{2} + \frac{2x}{3} - \frac{3x^2}{4} + \frac{4x^3}{5} - \ldots \tag{3.6}$$

so that

$$\int x\eta_1'(x)\,dx = -\frac{x^2}{2^2} + \frac{2x^3}{3^2} - \frac{3x^4}{4^2} + \frac{4x^5}{5^2} - \ldots \tag{3.7}$$

So that in (3.5) this gives

$$x\eta_2(x) = Ln(1+x) - \frac{x^2}{2^2} + \frac{2x^3}{3^2} - \frac{3x^4}{4^2} + \frac{4x^5}{5^2} - \ldots + k \tag{3.8}$$

Putting $x = 0$ gives $k = 0$ and then putting $x = 1$ yields

$$\eta(2) = Ln(2) + \frac{1}{4} - \frac{2}{9} + \frac{3}{16} - \frac{4}{25} + \ldots \tag{3.9}$$

Ideally the solution would be completed if the closed form of the secondary series in (3.9) could now be inserted. However, if this were done, the whole process collapses because, as with a polylogarithmic integral, self cancellation of the required result occurs. Since this is the case, the next step to further a solution is to introduce Euler' formula. To keep the analysis as simple as possible, the secondary series in (3.9) is split as follows

$$\begin{aligned}\eta(2) = Ln(2) + \frac{1}{4} + \frac{3}{16} + \frac{5}{36} + \frac{7}{64} \ldots + \frac{(2n-1)}{4n^2} + \ldots \\ - \left(\frac{2}{9} + \frac{4}{25} + \frac{6}{49} + \frac{8}{81} + \ldots + \frac{2n}{(2n+1)^2} + \ldots\right)\end{aligned} \tag{3.10}$$

Now, Euler's formula states that if t is an equation representing the tth term of a series and $S(t)$ is the partial sum to that term, then,

$$S(t) = \int t\,dn + \frac{t}{2} + \frac{1}{12}\frac{dt}{dn} - \frac{1}{720}\frac{d^3t}{dn^3} + \frac{1}{30240}\frac{d^5t}{dn^5} - \ldots \tag{3.11}$$

The validity of this relationship was provided by Euler in [4].

Thus employing Euler's formula on the first half of the secondary series above gives,

$$S_2(t) = \int \frac{(2n-1)}{4n^2 dn} dn + \frac{(2n-1)}{8n^2} + \frac{1}{12}\frac{d}{dn}\left(\frac{2n-1}{4n^2}\right) - \frac{1}{720}\frac{d^3}{dn^3}\left(\frac{2n-1}{4n^2}\right) + \frac{1}{30240}\frac{d^5}{dn^5}\left(\frac{2n-1}{4n^2}\right) \tag{3.12}$$

In (3.12)

$$\int \frac{(2n-1)}{4n^2 dn} dn = \frac{1}{2}Ln(n) + \frac{1}{4n} + C_2 \tag{3.13}$$

$$\frac{d}{dn}\left(\frac{2n-1}{4n^2}\right) = -\frac{1}{2}\left(\frac{n-1}{n^3}\right) \tag{3.14}$$

$$\frac{d^3}{dn^3}\left(\frac{2n-1}{4n^2}\right) = \frac{3(2-n)}{n^5} \tag{3.15}$$

and

$$\frac{d^5}{dn^5}\left(\frac{2n-1}{4n^2}\right)=\frac{60(3-n)}{n^7} \tag{3.16}$$

So that in (3.12) these give

$$S_2(t)=\frac{1}{2}Ln(n)+\frac{1}{4n}-\frac{1}{24}\left(\frac{n-1}{n^3}\right)-\frac{1}{240}\left(\frac{2-n}{n^5}\right)+\frac{1}{504}\left(\frac{3-n}{n^7}\right)+C_2 \tag{3.17}$$

and this reduces to

$$S_2(t)=\frac{1}{2}Ln(n)+\frac{840n^6-280n^5+70n^4+7n^3-14n^2-3\tfrac{1}{3}n+10}{1680n^7}+C_2 \tag{3.18}$$

The constant of integration in (3.18), C_2, is given by the partial sum of the first half of the secondary series in (3.10), at a value of n that produces any desired level of precision. Therefore C_2 becomes

$$C_2(n)=\sum_{q=1}^{n}\frac{2q-1}{4q^2}-\frac{1}{2}Ln(n)-\frac{840n^6-280n^5+70n^4+7n^3-14n^2-3\tfrac{1}{3}n+10}{1680n^7} \tag{3.19}$$

When an identical process is applied to the second half of the secondary series in (3.10), (to the third differential), the results are

$$S_2'(t)=\frac{1}{2}Ln(2n+1)-\frac{3840n^5+5760n^4+3200n^3+640n^2-82n-37}{240(2n+1)^5}+C_2' \tag{3.20}$$

So that

$$C_2'(n)=\sum_{q=1}^{n}\frac{2q}{(2q-1)^2}-\frac{1}{2}Ln(2n+1)+\frac{3840n^5+5760n^4+3200n^3+640n^2-82n-37}{240(2n+1)^5} \tag{3.21}$$

It is important to note that the n in $S_2(t)$ and $S'_2(t)$ is the normal series term number while the n in the constants $C_2(n)$ and $C'_2(n)$ is a term number specifically selected such that the precision of the result is set to a desired level. The two therefore have separate meanings and cannot be equated.

Combining the above results into (3.10) gives

$$\eta(2)=Ln(2)+S_2(t)-S_2'(t) \tag{3.22}$$

In $S_2(t)$ and $S_2'(t)$ when $n >> 1$, they become

$$\begin{aligned} S_2(t)&=\frac{1}{2}Ln(n)+C_2(n)\\ S_2'(t)&=\frac{1}{2}Ln(2n)-\frac{1}{2}+C_2'(n) \end{aligned} \tag{3.23}$$

So that (3.22) becomes

$$\eta(2) = Ln(2) + \frac{1}{2}Ln(n) - \frac{1}{2}Ln(2n) + \frac{1}{2} + C_2(n) - C_2^{\prime}(n)$$
$$= \frac{1}{2}\{1 + Ln(2)\} + C_2(n) - C_2^{\prime}(n) \tag{3.24}$$

with $C_2(n)$ and $C_2^{/}(n)$ as given above in (3.19) and (3.21). Eq(3.24) is the logarithmic version of the closed form of η(2).

Calculation for n = 10 yields the following results

$$C_2(10) = -0.122625685$$
$$C_2^{\prime}(10) = -0.098518868 \tag{3.25}$$

To give

$$\eta(2) = \frac{1}{2}\{1 + Ln(2)\} - 0.122625685 + 0.098518868$$
$$= 0.822466773 \tag{3.26}$$

and this result is within +2.59E-7 of the correct value evaluated to 15 decimal places. To increase precision, n in $C_2(n)$ and $C_2^{/}(n)$ would need to be increased.

<u>3.2 The Closed Form of Eta(3) and Zeta(3).</u>

Repeating (3.5)

$$x\eta_2(x) = Ln(1+x) - \int x\eta_1^{\prime}(x)\,dx \tag{3.27}$$

Differentiate and re-arrange to get

$$\eta_2(x) = \frac{1}{1+x} - x\eta_1^{\prime} - x\eta_2^{\prime} \tag{3.28}$$

and re-integrate

$$x\eta_3(x) = Ln(1+x) - \int x\eta_1^{\prime}\,dx - \int x\eta_2^{\prime}\,dx \tag{3.29}$$

Inserting from (3.5)

$$x\eta_3(x) = x\eta_2(x) - \int x\eta_2^{\prime}\,dx \tag{3.30}$$

and evaluation and substitution for the integral gives

$$x\eta_3(x) = x\eta_2(x) + \frac{x^2}{2^3} - \frac{2x^3}{3^3} + \frac{3x^4}{4^3} - \frac{4x^5}{5^3} + \ldots \tag{3.31}$$

where the constant of integration is zero. Splitting the secondary series and putting x = 1 then yields

$$\eta(3)=\eta(2)+\frac{1}{2^3}+\frac{3}{4^3}+\frac{5}{6^3}+\frac{7}{8^3}\ldots+\frac{(2n-1)}{8n^3}+\ldots$$
$$-\left(\frac{2}{3^3}+\frac{4}{5^3}+\frac{6}{7^3}+\frac{8}{9^3}+\ldots+\frac{2n}{(2n+1)^3}+\ldots\right) \tag{3.32}$$

Application of Euler's formula up to the third differential to both halves of the secondary series then gives

$$S_3(t)=\frac{-24n^5+18n^4-10n^3+3n^2+{}^4\!/_5\,n-1}{96n^6}+C_3 \tag{3.33}$$

for the first half and

$$S_3^{/}(t)=\frac{-48n^5-84n^4-68n^3-30n^2-{}^{22}\!/_5\,n+{}^{29}\!/_{20}}{3(2n+1)^6}+C_3^{/} \tag{3.34}$$

for the second half. The constants of integration are then

$$C_3(n)=\sum_{q=1}^{n}\frac{2q-1}{8q^3}-\left(\frac{-24n^5+18n^4-10n^3+3n^2+{}^4\!/_5\,n-1}{96n^6}\right)$$

and (3.35)

$$C_3^{/}(n)=\sum_{q=1}^{n}\frac{2q}{(2q+1)^3}-\left(\frac{-48n^5-84n^4-68n^3-30n^2-{}^{22}\!/_5\,n+{}^{29}\!/_{20}}{3(2n+1)^6}\right)$$

Substitution back into (3.32) then gives

$$\eta(3)=\eta(2)+S_3(t)-S_3^{/}(t) \tag{3.36}$$

and when $n\rightarrow\infty$, $S_3(t)$ and $S_3^{/}(t)\rightarrow C_3(n)$ and $C_3^{/}(n)$ respectively and (3.36) finally becomes

$$\eta(3)=\frac{\pi^2}{12}+C_3(n)-C_3^{/}(n) \tag{3.37}$$

Calculation for n = 10 yields the following results

$$C_3(10)=0.260976553$$
$$C_3^{/}(10)=0.181900758 \tag{3.38}$$

to finally give

$$\eta(3)=0.901542828 \tag{3.39}$$

which is within 1.51E-7 of the correct value evaluated to 15 decimal places. Consequently

$$\zeta(3)=\frac{2^2}{2^2-1}\eta(3)=\frac{\pi^2}{9}+\frac{4}{3}\left\{C_3(10)-C_3'(10)\right\}=1.202057104 \tag{3.40}$$

To a similar level of precision.

3.3 The Closed Forms of Eta(5) and Zeta(5).

To determine these it is necessary to skip over the Eta(4) integration. Therefore repeating (3.30)

$$x\eta_3(x)=x\eta_2(x)-\int x\eta_2'\,dx \tag{3.41}$$

Differentiating, re-arranging and re-integrating then gives

$$x\eta_4(x)=x\eta_3(x)-\int x\eta_3'\,dx \tag{3.42}$$

and repeating

$$x\eta_5(x)=x\eta_4(x)-\int x\eta_4'\,dx \tag{3.43}$$

All constants of integration are zero and (3.43) becomes, after evaluating the integral, putting x = 1 and splitting the secondary series

$$\begin{aligned}\eta(5)=\eta(4)&+\frac{1}{2^5}+\frac{3}{4^5}+\frac{5}{6^5}+\frac{7}{8^5}\ldots+\frac{(2n-1)}{32n^5}+\ldots\\&-\left(\frac{2}{3^5}+\frac{4}{5^5}+\frac{6}{7^5}+\frac{8}{9^5}+\ldots+\frac{2n}{(2n+1)^5}+\ldots\right)\end{aligned} \tag{3.44}$$

Application of Euler's formula up to the third differential then yields for the first half of the secondary series

$$S_5(t)=\frac{-48n^5+90n^4-84n^3+30n^2+8n+71}{2304n^8}+C_5 \tag{3.45}$$

So that

$$C_5(n)=\sum_{q=1}^{n}\frac{2q-1}{32q^5}-\left(\frac{-48n^5+90n^4-84n^3+30n^2+8n+71}{2304n^8}\right) \tag{3.46}$$

and for the second half of the secondary series

$$S_5'(t)=\frac{-128n^5-80n^4-64n^3-56n^2+56n-19}{24(2n+1)^8}+C_5' \tag{3.47}$$

So that

$$C_5'(n)=\sum_{q=1}^{n}\frac{2q}{(2q+1)^5}-\left(\frac{-128n^5-80n^4-64n^3-56n^2+56n-19}{3(2n+1)^6}\right) \tag{3.48}$$

and therefore in (3.43)

$$\eta(5)=\eta(4)+S_5(t)-S_5^{/}(t) \tag{3.49}$$

and when $n \to \infty$, $S_5(t)$ and $S_5^{/}(t) \to C_5(n)$ and $C_5^{/}(n)$ respectively to finally yield

$$\eta(5)=\frac{7\pi^4}{720}+C_5(n)-C_5^{/}(n) \tag{3.50}$$

Calculation for n = 10 then yields the following results

$$\begin{aligned} C_5(10)&=0.035241211 \\ C_5^{/}(10)&=0.010317192 \end{aligned} \tag{3.51}$$

To give

$$\eta(5)=0.972119772 \tag{3.52}$$

Which is within 1E-9 of the correct result evaluated to 15 decimal places. Finally

$$\zeta(5)=\frac{2^4}{2^4-1}\eta(5)=\frac{8\pi^4}{675}+\frac{16}{15}\left\{C_5(10)-C_5^{/}(10)\right\}=1.036927757 \tag{3.53}$$

To a similar level of precision.

<u>4.0 Generalisation.</u>

Full generalisation is not possible because Euler's formula is not conducive to this. The following general formulae, where $m > 2$, do however provide some alleviation to excessive computation.

$$\eta(m)=\eta(m-1)+C_m(n)-C_m^{/}(n) \tag{4.1}$$

where taken to the third differential in Euler's formula

$$C_m(n)=\sum_{q=1}^{n}\frac{(2q-1)}{2^m q^m}-\left[\begin{array}{l}\dfrac{1}{2^m(m-1)\,n^m}\left\{1-\dfrac{2(m-1)\,n}{m-2}\right\}+\dfrac{(2n-1)}{2^{(m+1)}n^m}-\dfrac{1}{12}\left\{\dfrac{2(m-1)\,n}{2^m\,n^{(m+1)}}\right\} \\ +\dfrac{m(m+1)}{720}\left\{\dfrac{2n(m-1)-(m-2)}{2^m\,n^{(m+3)}}\right\}\end{array}\right] \tag{4.2}$$

and

$$C_m^{/}(n)=\sum_{q=1}^{n}\frac{2q}{(2q+1)^m}-\left[\begin{array}{l}\dfrac{-n}{(m-1)(2n+1)^{(m-1)}}-\dfrac{1}{2(m-1)(m-2)(2n+1)^{(m-2)}}+\dfrac{n}{(2n+1)^m} \\ +\left\{\dfrac{1-2(m-1)\,n}{6(2n+1)^{(m+1)}}-\dfrac{2}{15}\left\{\dfrac{3(m+1)-2(m^2-1)}{(2n+1)^{(m+3)}}\right\}\right\}\end{array}\right] \tag{4.3}$$

If m here is odd, then since $\eta(m - 1)$ is the value for an even exponent, for which the closed form values are already known, the solution for $\eta(m)$ for any odd exponent can therefore be determined from a single application of the relationships above.

5.0 Discussion of Results.

Clearly the results obtained here cannot be considered as complete closed forms, because they still rely upon the summation of a small number of terms of the secondary series to determine the constants of integration in the application of Euler's formula. Hence the use of the word 'pseudo' in the title of this paper. Accordingly it could be said that the same results would be more easily obtained by simply subtracting $\eta(m - 1)$ from $\eta(m)$ to obtain a value k and then say that

$$\eta(m) = \eta(m-1) + k \tag{5.1}$$

While such an exercise is perfectly valid, it requires pre-knowledge of both $\eta(m - 1)$ and $\eta(m)$, and does not provide any substance to the constant k, it is merely a number. Similarly, it could be suggested that Euler's formula be applied directly to $\eta(m)$. Again while that would be quite valid it would simply obtain $\eta(m)$ as a number. The relationship with $\eta(m - 1)$ would not be produced.
Consequently, while the results obtained here are not fully closed forms, they are superior to the above suggestions in that the relationship between $\eta(m)$ and $\eta(m - 1)$, has been analytically derived as has the $C_m(n)$ and $C'_m(n)$ constants. They do also provide the opportunity for further study, in that if the closed forms of the secondary series could be determined in a suitable form, the solutions to the series studied here would become complete.

There are two means available to increase the level of precision if desired. They are (i) increase the value of n in calculating the values of $C_m(n)$ and $C'_m(n)$, and/or, (ii) increase the number of differential terms in Euler's formula.

Finally, although emphasis has been placed on determining closed forms for these series with odd exponents, it is clear that the method is equally applicable to the series with even exponents, to obtain the logarithmic version of their closed forms. This is illustrated in the Appendix.

6.0 Conclusions.

The primary difficulty in determining fully closed forms for any infinite series of these types with odd exponents, is that such exponents are either prime numbers, multiples of prime numbers or powers of prime numbers. Therefore, unlike the even exponents, there is no one single common factor. Accordingly, it is possible that such infinite series do not possess fully closed forms in terms of elementary functions. However, if they do, it is likely that they will have to be analysed in terms of their exponents according to each individual prime and their multiples and powers.

Appendix A.

The Logarithmic Version of the Closed Forms for All Values of the Exponent.

It should be noted that to obtain this version for some value *m* of the exponents, it is necessary to determine this version for all (*m* - 1) values as they form a ladder relationship. Also in doing so the results will contain a proliferation of the constants of integration from the application of Euler's formula. Thus starting from the base formula for the Eta series.

$$\eta(1) = Ln(2) \tag{A.1}$$

then

$$\eta(2) = \frac{1}{2}\{1 + Ln(2)\} + C_2(n) - C_2'(n) \tag{A.2}$$

and then

$$\begin{aligned}\eta(3) &= \eta(2) + C_3(n) - C_3'(n) \\ &= \frac{1}{2}\{1 + \boldsymbol{ln}(2)\} + \sum_{q=2}^{3}\{C_q(n) - C_q'(n)\}\end{aligned} \tag{A.3}$$

So that

$$\eta(m) = \frac{1}{2}\{1 + \boldsymbol{ln}(2)\} + \sum_{q=2}^{m}\{C_q(n) - C_q'(n)\} \tag{A.4}$$

and when $m \to \infty$

$$\eta(\infty) = \frac{1}{2}\{1 + \boldsymbol{ln}(2)\} + \sum_{q=2}^{\infty}\{C_q(n) - C_q'(n)\} = 1 \tag{A.5}$$

Consequently

$$\sum_{q=2}^{\infty}\{C_q(n) - C_q'(n)\} = \frac{1}{2}\{1 - \boldsymbol{ln}(2)\} = 0.153426409 \tag{A.6}$$

and is independent of *n*.

References.

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