A Geometrical/Empirical Determination

of the Circumference of an Ellipse.

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Abstract.

This paper presents a geometrical/empirical method to determine the circumference of any ellipse. The method requires knowledge of only two descriptive parameters of the ellipse and provides a potential accuracy of better than 0.0014% for eccentricities of $0 < e < 1$.

Contents

1.0 Introduction.

It is well known that the determination of the perimeter of an ellipse reults in the elliptic integral of the second kind, viz.

$$p = 4a\int_0^{\pi/2} \left(1 - e^2 \sin^2 \phi\right) d\phi \tag{1.1}$$

and that this integral cannot be evaluated analytically but must be binomially expanded and then integrated term by term.

It is also well known that every ellipse can be created by deforming a circle. Such a circle may be referred to as a Generating Circle and if the deformation takes place without stretching the perimeter, then the circumference of the ellipse will be identical to that of the generating circle.

It is the purpose of this paper, using a geometrical/empirical method to establish a relationship between a main parameter of any ellipse and the radius of its generating circle so that the above objective can be realised.

2.0 Nomenclature.

The following nomenclature will be used in this paper.

(i) r – represents the radius of the generating circle.
(ii) a – represents the semi-major axis of the ellipse.
(iii) b – represents the semi-minor axis of the ellipse.
(iv) e – represents the eccentricity of the ellipse.
(v) L – represents the length of a quadrant after the first deformation of the generating circle along the x axis.
(vi) l – represents the amount by which r is elongated along the x axis after the first deformation of the generating circle.
(vii) d_1 – represents the length of the second deformation of the generating circle along the y axis.
(viii) d_2 – represents the length of the third deformation of the generating circle along the y axis.
(ix) p –represents the perimeter of the generating circle and its subsequently deformed shapes.
(x) ABCD –represent the east-west and north-south extremities of the generating circle and its subsequently deformed shapes.

3.0 Description of the Method and Determination of the Algorithm.

3.1 The Geometric Process.

Starting with any generating circle

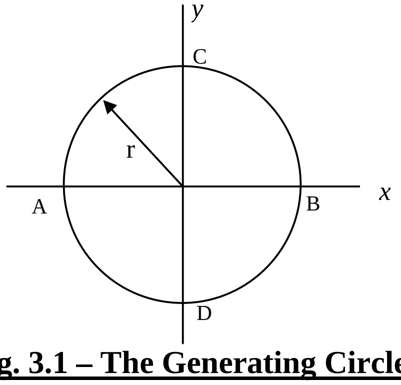

Fig. 3.1 – The Generating Circle.

where the perimeter is

$$p = 2\pi r \tag{3.1}$$

The procedure to generate a specific ellipse is as follows. First, keeping points C and D fixed, stretch A and B along the *x* axis so that the curves AC, CB, BD and DA become straight Lines.

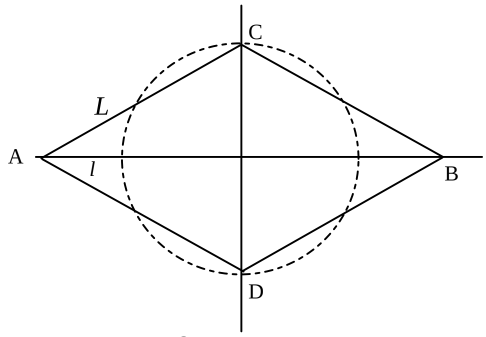

Fig. 3.2 – First Deformation Along the *x* Axis.

The circumference of the diamond is 4*L* and therefore for this first deformation.

$$4L = 2\pi r \tag{3.2}$$

so that

$$L = \frac{\pi r}{2} \tag{3.3}$$

and also with

$$L = \left\{ r^2 + (r + l)^2 \right\}^{1/2} \tag{3.4}$$

then

$$l = r\left\{ \left(\frac{\pi^2}{4} - 1 \right)^{1/2} - 1 \right\} \tag{3.5}$$

This relationship is required when dealing with the special cases in Section 4.3.

Now vary CD either up or down by an amount d_1 while maintaining the length *L*, as in (3.3) constant. This means that the points A and B must slide along the *x* axis accordingly. This process is necessary to cover the complete range of ellipse eccentricity. However, there are limits to the value of d_1 which must be adhered to, i.e. $-r \le e \le l$.

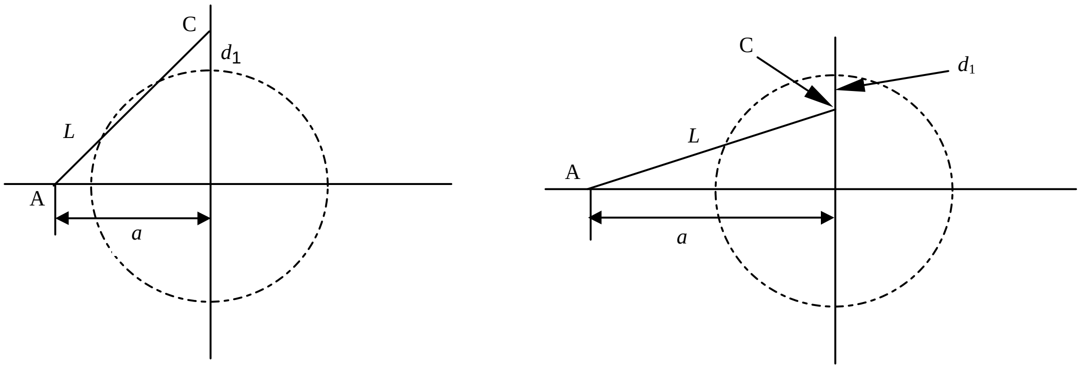

Fig. 3.3 – Second Deformation along the *y* Axis, d_1 +ve and –ve,
(Top Left Quadrant Only Shown).

This establishes the ellipse parameter *a* as

$$a = \left\{ \frac{\pi^2 r^2}{4} - \left(r \pm d_1\right)^2 \right\}^{1/2} \tag{3.6}$$

Finally, keeping A and B fixed, depress CD by an amount d_2 so that each quadrant of the diamond is forced into the shape of a quadrant of an ellipse. Note however, for each value of *r* there is only one value of d_2 that satisfies this criteria and which is established by iteratively varying d_2 until the perimeter of the ellipse, as determined from the ellipse integral of (1.1), expanded to the desired precision, is equal to the perimeter of the generating circle. This establishes the ellipse parameter *b* as

$$b = r - d_1 - d_2 \tag{3.7}$$

This process generates an ellipse from its specific generating circle and thus provides its perimeter.

3.2 Determination of the Algorithm.

For a random ellipse, it is necessary to establish an empirical relationship between a selected parameter of the ellipse and the radius of its generating circle for the ellipse perimeter to be determined.

The elliptical parameter chosen to establish the relationship to *r* was via a dummy variable *k* as in

$$k = \frac{a - r}{a} \tag{3.8}$$

To start the process, the elliptical integral of (1.1) was first expanded into a suitable number of terms as in Appendix A. This was also required to permit the accuracy of the ensuing algorithm to be measured. Now for a suitable value of *r*, (10 was used here), the procedure of Section 3.1 was performed for a large number of +ve and –ve values of d_1, to ensure the full range of *e* was covered. For each value of d_1, d_2 was iteratively varied until the perimeter result of the above elliptical integral expansion, calculated using the value of *a* as determined from (3.6), was equal to the perimeter of the generating circle. At each result, as well as *a* being determined from (3.6), so also was *k* from (3.8) and *b* from (3.7), thus enabling *e* to be determined from the usual formula

$$e = \left(1 - \frac{b^2}{a^2}\right)^{1/2} \tag{3.9}$$

The full set of results for this exercise are shown tabulated below.

d_1	a	d_2	b	e	k
2.10	10.016491902220	-2.083494487401	9.983494487401	0.081103419238	0.001646475
2.00	10.136079618237	-1.862988090156	9.862988090156	0.230562661034	0.013425271
1.75	10.424855395987	-1.315917094512	9.565917094512	0.397489566243	0.040754080
1.50	10.700005141458	-0.774555575237	9.274555575237	0.498689885891	0.065421010
1.25	10.962554904183	-0.238589115723	8.988589115723	0.572455762013	0.087803884
1.00	11.213389765242	0.292256996528	8.707743003472	0.630057259265	0.108209006
0.75	11.453279444213	0.818224292252	8.431775707748	0.676776251627	0.126887626
0.5	11.682898186120	1.339526392820	8.160473607180	0.715612625698	0.144048006
0.25	11.902840418456	1.856353555839	7.893646444161	0.748465844365	0.159864398
0.00	12.113633229846	2.368876998341	7.631123001659	0.776626333943	0.174483839
-0.25	12.315746425907	2.877253394394	7.372746605606	0.801015273254	0.188031350
-0.75	12.695574426832	3.882148461736	6.867851538264	0.841045605714	0.212323943
-1.00	12.874009089139	4.378952694867	6.621047305133	0.857612937072	0.223241188
-1.25	13.045214065980	4.872191094588	6.377808905412	0.872339675257	0.233435347
-1.50	13.209470467329	5.362022180170	6.137977819830	0.885486585042	0.242967383
-1.75	13.367034451487	5.848617846086	5.901382153914	0.897267270141	0.251890908
-2.00	13.518140035790	6.332166066929	5.667833933071	0.907858742407	0.260253262
-2.25	13.663001501399	6.812872902722	5.437127097278	0.917409148646	0.268096399
-2.50	13.801815461280	7.290964041125	5.209035958875	0.926043528900	0.275457636
-3.00	14.062009459079	8.240308776935	4.759691223065	0.940974039788	0.288864082
-4.00	14.516890508206	10.120153283748	3.879846716252	0.963623260887	0.311147246
-5.00	14.890940535347	11.997059119897	3.002940880103	0.979455118219	0.328450747
-6.00	15.190131995056	13.921084920607	2.078915079393	0.990590457991	0.341677873
-7.00	15.418823237434	16.102291915000	0.897708085000	0.998303685695	0.351442075
-7.25	15.465368085734	16.937603610350	0.312396389650	0.999795964469	0.353393987

Table 3.1 – Results of the Algorithm Determing Process.

k was then plotted against e and curve fitted to produce the following 10^{th} order polynomial algorithm.

$$k = \alpha + \beta e + \chi e^2 + \delta e^3 + \varepsilon e^4 + \phi e^5 + \gamma e^6 + \eta e^7 + \iota e^8 + je^9 + \kappa e^{10} \qquad (3.10)$$

where the coefficients are

$\alpha =$	1.5460847E-08
$\beta =$	-6.1532324E-02
$\chi =$	1.9421389E+00
$\delta =$	-1.7450132E+01
$\varepsilon =$	9.4169113E+01
$\phi =$	-3.0192772E+02
$\gamma =$	6.0864890E+02
$\eta =$	-7.8059437E+02
$\iota =$	6.1971507E+02
$\varphi =$	-2.7822962E+02
$\kappa =$	5.4141807E+01

Table 3.2 – Coefficients of the Polynomial Algorithm.

Thus, for any ellipse where only two of a, b or e are known, (e determined from (3.9) as necessary), k can be calculated from (3.10) which in turn allows the radius of the generating circle to be ascertained from (3.8) as

$$r = a(1-k) \qquad (3.11)$$

so obtaining the perimeter of the ellipse via (3.1).

4.0 Application to Three Examples.

4.1 For $e < 0.0001$.

Let a be 20 and b be 19.99999992. Then via (3.9) e = 0.000089442719. From the algorithm (3.10) for this value of e, k is determined to be 0.0000054726, which in turn via (3.11) produces a value of r of 20.0001094527, and thus an elliptic circumference of 125.6643938549. The other parameters involved in this example are

d_1 = 4.2273799485 from (3.6)

and

d_2 = 4.2272704158 from (3.7)

4.2 For $e = 0.5$.

Again let a = 20. From the algorithm, (3.10), with this value of e, k =0.0657935619 which via (3.11) produces an r of 18.6841287628. and thus an elliptic circumference of 117.39584332. The other parameters involved in this example are

d_1 = 2.7951985498 from (3.6)

and

d_2 = -1.4315778626 from (3.7)

and

b = 17.3205080756888 from (3.9).

4.3 For $e > 0.9999$.

Let a be 300 and b = 0.1 so that e =0.999999944444. From the algorithm, (3.10) this gives k as 0.3536545184, which in turn via (3.11) produces an r of 193.9036444795 and an elliptic perimeter of 1218.332530002. The other parameters involved in this example are

d_1 = -141.2644518202 from (3.6)

and

d_2 = 335.0680962997 from (3.7)

4.4 Two Special Cases, $e = 0$ and $e = 1$.

These two cases are somewhat trivial because the respective circumferences are intuitively known. They are however included to show that the method described produces exact results. The algorithm is not needed for either case because the results are obtained directly from (3.6).

(i) When e = 0, the shape being considered is the generating circle itself so that $a = b = r$. Subsequent to the first deformation along the x axis, thus obtaining the diamond shape of Fig. 3.2, insert $d_1 = l$ as per (3.5) into (3.6) to give

$$a = \left[\frac{\pi^2 r^2}{4} - \left\{ r + \left[\left(\frac{\pi^2}{4} - 1\right)^{1/2} - 1\right]\right\}^2\right] \qquad (4.1)$$

which educes to

$$a = r \qquad (4.2)$$

So that $$p = 2\pi a \qquad (4.3)$$

Note that this automatically sets $d_2 = -d_1$ from (3.7).

(ii) When $e = 1$, the deformation of the generating circle is down to a straight line along the x axis. In this case put $d_1 = -r$ in (3.6) so that it becomes

$$-r = \left(\frac{\pi^2 r^2}{4} - a^2\right)^{1/2} - r \qquad (4.4)$$

which reduces to

$$r = \frac{2a}{\pi} \qquad (4.5)$$

giving the 'circumference' as

$$p = 2\pi \bullet \frac{2a}{\pi} = 4a \qquad (4.6)$$

Note that this automatically sets $d_2 = 0$ from (3.7).

The interesting point here is the radius of the generating circle at $r = \frac{2a}{\pi}$.

Note that the full range of k is from (3.8)

When $e = 0$ and $r = a$, $\quad k = 0$

When $e = 1$ and $r = \frac{2a}{\pi}$, $\quad k = \frac{a - \frac{2a}{\pi}}{a} = 1 - \frac{2}{\pi} = 0.363380228$

and is a maximum.

5.0 Accuracy.

It is to be noted that the accuracy figures presented here are against the expanded elliptic integral of Appendix A where only 17 terms were included. As such they are largely a measure of the efficacy of the algorithm

e	*a*	*b*	Algorithm Error
0.0000000000	20.0000000000	20.0000000000	0.0000000000%
0.0000894427	20.0000000000	19.9999999200	0.0005474633%
0.0811034192	10.0164919022	9.9834944874	0.0000049889%
0.2305626610	10.1360796182	9.8629880902	-0.0000318903%
0.3974895662	10.4248553960	9.5659170945	0.0002890763%
0.4986898859	10.7000051415	9.2745555752	-0.0010327426%
0.5000000000	20.0000000000	17.3205080757	-0.0009654654%
0.5724557620	10.9625549042	8.9885891157	0.0013630481%
0.6300572593	11.2133897652	8.7077430035	0.0001365624%
0.6767762516	11.4532794442	8.4317757077	-0.0009652843%
0.7156126257	11.6828981861	8.1604736072	-0.0007404082%
0.7484658444	11.9028404185	7.8936464442	0.0001031538%
0.7766263339	12.1136332298	7.6311230017	0.0007562689%
0.8010152733	12.3157464259	7.3727466056	0.0009018854%
0.8410456057	12.6955744268	6.8678515383	0.0001607887%
0.8576129371	12.8740090891	6.6210473051	-0.0002551983%
0.8723396753	13.0452140660	6.3778089054	-0.0004864703%
0.8854865850	13.2094704673	6.1379778198	-0.0004912861%
0.8972672701	13.3670344515	5.9013821539	-0.0003000154%
0.9078587424	13.5181400358	5.6678339331	0.0000179002%
0.9174091486	13.6630015014	5.4371270973	0.0003823222%
0.9260435289	13.8018154613	5.2090359589	0.0007216916%
0.9409740398	14.0620094591	4.7596912231	0.0011365307%
0.9636232609	14.5168905082	3.8798467163	0.0006943024%
0.9794551182	14.8909405353	3.0029408801	-0.0003099636%
0.9905904580	15.1901319951	2.0789150794	-0.0004145898%
0.9983036857	15.4188232374	0.8977080850	0.0007357745%
0.9997959645	15.4653680857	0.3123963897	0.0011627515%
0.9999999444	300.0000000000	0.1000000000	0.0012276406%
1.0000000000	20.0000000000	0.0000000000	0.0000000000%
Largest Error			0.0013630481%
Average Error			0.0001499598%

Table 5.1 – Accuracy of the Algorithm Measured Against the Elliptic Integral Expansion, (17 Terms).

The numbers in blue are the results of the examples in Section 4.2. The main point of interest here is that the accuracy is very consistent across the whole range of *e*.
A comparison with other algorithms, as described in [1], [2], [3] and [4] has not been included in this paper because, as stated above only 17 terms of the expanded elliptic integral was used to construct the algorithm and other algorithms most probably will have been constructed with reference to a larger number of terms making the comparison unrealistic.

6.0 Conclusions.

To improve the accuracy of the method to both provide an enhanced algorithm and against which to measure algorithm efficacy would require taking a great many more terms in the expansion of the elliptic integral. For instance to obtain an accuracy of better than 10 decimal places for *e* up to 0.9865 would require some 500 terms. This would then make comparison with other algorithms viable.

The algorithm developed here does not exhibit a reduction in accuracy with increasing eccentricity, showing very consistent results across the whole range. This is because the algorithm was effectively derived directly from the generating circle and the expansion of the elliptic integral. This is also a major feature of the overall method in that selecting a specific generating circle it is possible to construct a very precise ellipse of any desired size and shape. Similarly, in the determination of the perimeter of a random ellipse from its three basic parameters, the method provides full details of its generating circle relationship. Also as a consequence of this it is believed that by taking a sufficiently large number of terms in the elliptic integral expansion together with a very high precision curve fitting technique to construct a superior algorithm, could provide an accuracy that was only largely limited by the irrationality of Pi. However, for extremely high values of e the number of terms in the expansion of the elliptic integral would most probably require main frame computing facilities.

APPENDIX A.

Details of the Elliptic Integral Expansion to Construct the Algorithm and Assess Accuracy.

The elliptic integral expansion used to construct the algorithm and assess accuracy was to 17 terms as shown below.

$$
\begin{aligned}
p/4\pi a = \; & +0.5 \\
& -0.125e^{2} \\
& -0.0234375e^{4} \\
& -0.009765625e^{6} \\
& -0.005340576e^{8} \\
& -0.003364563e^{10} \\
& -0.002313137e^{12} \\
& -0.001687646e^{14} \\
& -0.001285512e^{16} \\
& -0.001011745e^{18} \\
& -0.000816984e^{20} \\
& -0.000673506e^{22} \\
& -0.000564763e^{24} \\
& -0.000480382e^{26} \\
& -0.000413594e^{28} \\
& -0.000359827e^{30} \\
& -0.000315903e^{32}
\end{aligned}
$$

References.

[1] The Maths Forum, *Circumference of an Ellipse*, www.Mathforum.org.

[2] Shahram Zafary, *A Single Term Formula for Approximating the Circumference of an Ellipse*, www.Mathforum.org

[3] MathsisFun, *Perimeter of an Ellipse*, www.Mathsisfum.com

[4] Gerard P. Michon, *Perimeter of an Ellipse*, www.Numeriana.com.