GRAVITATION - A NEW THEORY

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Abstract

This paper presents a new relativistic theory of gravitation as an alternative to that represented in Albert Einstein's General Theory of Relativity. Initially, a new representation of the gravitational space-time continuum, designated the Relativistic Domain $\mathbf{D}_1$, is created utilising a system of linear co-ordinates. This Domain is subsequently shown to possess all the gravitational characteristics of the General Theory, and as observed in the Solar System and beyond. A new interpretation of the gravitational phenomenon is thus made, avoiding the problems associated with an induced curvature of the space-time continuum as is required in the General Theory.

1 Introduction.

The General Theory of Relativity, published by Albert Einstein in 1915/16, deals with the kinematics of motion of a free particle mass when under the exclusive influence of gravitation. Within the General Theory gravitation is not purported to be caused by an accelerative force, it is said to be caused by the presence of matter creating a distortion of the space-time continuum. The distortion is such that the continuum becomes curved in the direction of the gravitational source. A mass within this curved space-time, in motion under the sole influence of the source, then moves along a curved path, or geodesic, so gravitating towards it. The velocity of such motion increases with the increase in the degree of curvature as the source is approached. What is not clear in the General Theory however, is how a particle mass is caused to accelerate from rest, from any location within this curved space-time. Also, the mechanism causing the curvature is neither adequately defined nor mathematically described.

This paper provides an alternative approach to gravitation avoiding these difficulties by removing the need for a curved space-time continuum. In a manner identical to the analytical approach advanced in [1], the concept of an Existence Velocity within a Relativistic Domain is used to simplify, extend, and eventually re-define the gravitational phenomenon.

Initially, the precise definition of a linear Relativistic Space-Time Domain, $\mathbf{D}_1$, within which gravitation is subsequently shown to exist naturally, is effected. This permits the derivation of a simple expression for the cause of gravitational motion, defined as the Acceleration Potential of that Domain. This Potential, subsequent to the correlation of the Domain $\mathbf{D}_1$ with the Solar System, then enables an uncomplicated derivation of the major kinematic equations of gravitational motion, including those for a central orbit for a single particle mass. Utilising these results, an exact solution

of the equation of the orbit is then constructed for comparison with the approximate solutions of the General Theory, and with gravitational motion within the Solar System.
Where useful throughout the text, a physical interpretation of the results is given, as is frequent comparison with the applicable expressions of the General Theory.
In the interests of brevity, mathematical derivation has been kept as short as possible and only the main results presented. Also, familiarity with the general concept of a Relativistic Space-Time Domain and its main characteristic, Existence Velocity, as presented in [1], is assumed.

2 The Relativistic Space-Time Domain $\mathbf{D}_1$.

2.1 Definition.

The Relativistic Domain $\mathbf{D}_0$, as developed and shown in [1] to be equivalent to Pseudo-Euclidean Space-Time, is one in which gravitation exists in an artificially defined form only. A rigorously defined expression of gravitation, for a single gravitational source, requires that the Domain $\mathbf{D}_0$ is modified by the presence of the source, to produce a new Relativistic Domain, $\mathbf{D}_1$. The change is a simple one and the only differences between $\mathbf{D}_0$ and $\mathbf{D}_1$ are, firstly, a modification of the form of Existence Velocity, the central concept upon which such Domains are based, and secondly a consequential modification of the maximum theoretical spatial velocity attainable within the Domain. The new Domain is however still linear and does not exhibit any form of curvature.

Accordingly, $\mathbf{D}_1$ can be defined as a mutually orthogonal space-time of four linear dimensions, three of which $\mathbf{Y}_1$, $\mathbf{Y}_2$, and $\mathbf{Y}_3$, are spatial in nature, and the fourth, $\mathbf{X}_0$, is temporal and identical to the temporal dimension of $\mathbf{D}_0$. Time in $\mathbf{D}_1$ is represented by the parameter τ and, as a consequence of the new Domain's modified Existence Velocity, is different from the time t in $\mathbf{D}_0$. The Domain is such that it possesses a preferred spatial origin, the centre of the gravitational source, from which the radius vector magnitude to any random point B is

$$\sigma = \left(y_1^2 + y_2^2 + y_3^2\right)^{1/2} \tag{2.1}$$

where y_1 , y_2 and y_3 are each a distance along the respective spatial axes from the origin. σ has been chosen to represent the radius vector magnitude in $\mathbf{D}_1$ to separately identify it from the same parameter, r, in $\mathbf{D}_0$.
All spatial-temporal points that exist within $\mathbf{D}_1$ must, at all times, possess a characteristic Existence Velocity, the magnitude of which, for the point B, is defined to be the resultant of all four velocities along the co-ordinate axes of $\mathbf{D}_1$ and, may therefore be expressed as

$$V = \left(\dot{y}_1^2 + \dot{y}_2^2 + \dot{y}_3^2 + \dot{x}_p^2\right)^{1/2} = cu \tag{2.2}$$

where the $\dot{y}_{\#}$are the spatial axial velocities of the point and $\dot{x}_p$ is its temporal velocity. The parameter c is a velocity constant numerically equal to the magnitude of Existence Velocity in $\mathbf{D}_0$ and u is initially defined to be an arbitrary dimensionless function of σ.

Finally, the maximum theoretically attainable spatial velocity in $\mathbf{D}_1$, designated Spatial Terminal Velocity, is defined as follows. For motion purely along a radius vector, Spatial Terminal Velocity is defined to be equal to cu. For purely circular motion in any plane about the origin, it is defined to be equal to the velocity constant c. This difference exists because of the purely radial nature of gravitation.

2.2 Existence Within $\mathbf{D}_1$

The Spatial-Temporal Existence Velocity Vector $\mathbf{V}$, for the random point B within the Domain is determined as follows.
The spatial-temporal position of the point B with respect to the spatial centre of the gravitational source and some chosen temporal reference will be

$$\mathbf{S} = \boldsymbol{i}y_1 + \boldsymbol{l}y_2 + \boldsymbol{k}y_3 + \boldsymbol{j}x_p \tag{2.3}$$

where the $y_{\#}$ are each a distance along the three spatial axes $\mathbf{Y}_1$, $\mathbf{Y}_2$ and $\mathbf{Y}_3$ for which the $\boldsymbol{i}$, $\boldsymbol{l}$ and $\boldsymbol{k}$ are normal unit vectors. The term x_p is a distance along the temporal axis for which $\boldsymbol{j}$ is the unit vector with a magnitude of $\sqrt{-1}$.

From (2.1), Eq.(2.3) may be rewritten as

$$\mathbf{S} = \sigma\boldsymbol{n} + \boldsymbol{j}x_p \tag{2.4}$$

where $\boldsymbol{n}$ is a radial unit vector.
For planar motion, the velocity of this point is defined by differentiating (2.4) with respect to the time τ thus

$$\mathbf{V} = \dot{\sigma}\boldsymbol{n} + u\omega\sigma\boldsymbol{t} + \boldsymbol{j}\dot{x}_p \tag{2.5}$$

where $\mathbf{V} = {}^{d\mathbf{S}}/_{d\tau}$ and $\omega = \dfrac{d\phi}{d\tau}$ and is the angular rate of the point B.
Note that (2.5) involves the differential of the unit vector $\boldsymbol{n}$ thus

$$\frac{d\boldsymbol{n}}{d\tau} = \frac{d\boldsymbol{n}}{d\varphi}\frac{d\varphi}{d\tau} = u\omega\boldsymbol{t} \tag{2.6}$$

within which

$$\frac{d\boldsymbol{n}}{d\varphi} = u\boldsymbol{t} \tag{2.7}$$

and therefore similarly

$$\frac{d\boldsymbol{t}}{d\varphi} = -u\boldsymbol{n} \tag{2.8}$$

Relationships (2.7) and (2.8) occur because of the different Spatial Terminal Velocities in the radial, $\boldsymbol{n}$, and radial normal, $\boldsymbol{t}$, directions. Proof of the above relationships is presented in Appendix F.
Taking the magnitude of (2.5) gives, after invoking the characteristic of existence in $\mathbf{D}_1$ via the insertion of (2.2)

$$|\mathbf{V}| = V = cu = \left(\dot{\sigma}^2 + u^2\omega^2\sigma^2 + \dot{x}_p^2\right)^{1/2} \tag{2.9}$$

from which

$$\dot{x}_p = cu\left(1 - \frac{\dot{\sigma}^2}{c^2u^2} - \frac{\omega^2\sigma^2}{c^2}\right)^{1/2} \tag{2.10}$$

which when re-inserted into (2.5) yields

$$\mathbf{V} = \dot{\sigma}\boldsymbol{n} + u\omega\sigma\boldsymbol{t} + \boldsymbol{j}cu\left(1 - \frac{\dot{\sigma}^2}{c^2u^2} - \frac{\omega^2\sigma^2}{c^2}\right)^{1/2} \tag{2.11}$$

Eq.(2.11) is the Existence Velocity of the random point B in the Relativistic Space-Time Domain $\mathbf{D}_1$. This expression will be used in the next Section to develop the kinematics and kinetics of gravitational motion in $\mathbf{D}_1$ which will then be shown to be the natural state of existence in that Domain. Before that however, it is useful to note three other important characteristics of $\mathbf{D}_1$.

2.3 The Time τ in $\mathbf{D}_1$

The first concerns the time τ in $\mathbf{D}_{1.}$ From (2.11) when $\dot{\sigma}$ and ω are both zero, motion exists only along the temporal axis of $\mathbf{D}_1$ so that the temporal velocity of a spatially stationary point in $\mathbf{D}_1$ is

$$\frac{dx_0}{d\tau} = cu \tag{2.12}$$

and therefore an element of time in $\mathbf{D}_1$may be defined by the relationship

$$d\tau = \frac{dx_0}{cu} \tag{2.13}$$

and is therefore a function of spatial position from the origin by virtue of the fact that u is a function of σ.

2.4 The Proper Time in $\mathbf{D}_1$

The second point concerns the proper time of the point B in $\mathbf{D}_1$, i.e. the time measured by an observer located with the point B. Inserting (2.12) into (2.10) and re-arranging gives

$$\frac{dx_p}{dx_0} = \left(1 - \frac{\dot{\sigma}^2}{c^2u^2} - \frac{\omega^2\sigma^2}{c^2}\right)^{1/2} \tag{2.14}$$

using (2.13) to rewrite the LHS of (2.14) then gives

$$\frac{d\tau_p}{d\tau} = \left(1 - \frac{\dot{\sigma}^2}{c^2u^2} - \frac{\omega^2\sigma^2}{c^2}\right)^{1/2} \tag{2.15}$$

where

$$d\tau_p = \frac{dx_p}{cu} \tag{2.16}$$

where $^{d\tau_p}/_{d\tau}$ is the temporal rate, and τ_p the proper time of the point B in $\mathbf{D}_1$.

2.5 Temporal Significance of the Function u.

Although u has been defined as a non-dimensional function of the spatial variable σ, its appearance in the temporal components of the above relationships has a special significance in that it relates time in $\mathbf{D}_1$ to that in $\mathbf{D}_0$, (Pseudo-Euclidean Space-Time).

A spatially stationary point in$\mathbf{D}_1$, with a temporal velocity given by (2.12) would, in an element of time dt in $\mathbf{D}_0$ move an element of distance dx_0 along the temporal axis, given by

$$dx_0 = cudt \tag{2.17}$$

Therefore in $\mathbf{D}_{0,}$the proper time of such a point, i.e. the proper time of $\mathbf{D}_1$ would be

$$d\tau = \frac{dx_0}{c} = udt \tag{2.18}$$

so that

$$\frac{d\tau}{dt} = u \tag{2.19}$$

and u is therefore a measure of the temporal rate of $\mathbf{D}_1$with respect to $\mathbf{D}_0$ and, for future reference, it is noted that it must therefore possess a positive sign. Also, it is clear that because u is a function of σ, the temporal rate of $\mathbf{D}_1$ is a variable

dependent upon radial distance from the centre of the gravitational source, i.e. $\mathbf{D}_1$ exhibits spatially dependent temporal dilatation, as is also evident from (2.13).
The relationship between the respective spatial axes of the two domains depends upon the characteristics of u and will be developed subsequent to the determination of the precise nature of this function in Section 4.

3 Gravitational Planar Motion in $\mathbf{D}_1$.

3.1 The Accelerative Force of Gravitation.

It was shown in [1] that in the Relativistic Domain $\mathbf{D}_0$, (Pseudo Euclidean Space-Time), a change in the Existence Momentum of a mass could only be effected by the application of an accelerative force. Indeed, for spatial motion of a mass to exist in any Relativistic Domain, including one exhibiting gravitation, it is firmly believed that it can only result from the application of such a force. To cause gravitational motion therefore, if an accelerative force is not artificially applied, then it must be generated within the mass itself as a result of interaction with the characteristics of the Domain. Assuming the gravitational effect in $\mathbf{D}_1$ to be a purely spatial radial one, this internally generated gravitational force can be determined for any gravitating mass, by comparing the spatial variation of its total energy, with the temporal variation of its Existence Momentum. First, consider the variation of Existence Momentum with time. For planar motion it is derived as follows.

If m is the energy mass of a particle possessing free planar motion in $\mathbf{D}_1$, then, from (2.11), its Existence Momentum will be:-

$$\mathbf{M} = m\left\{\dot{\sigma}\,\boldsymbol{n} + u\omega\sigma\,\boldsymbol{t} + \boldsymbol{j}\,\left(c^2u^2 - \dot{\sigma}^2 - u^2\omega^2\sigma^2\right)^{1/2}\right\} \tag{3.1}$$

Differentiating (3.1) with respect to τ gives the time rate of change of $\mathbf{M}$ in $\mathbf{D}_1$ as :-

$$\begin{aligned}\frac{d\mathbf{M}}{d\tau} = &\ \left\{\dot{m}\,\dot{\sigma} + m\,\left(\ddot{\sigma} - u^2\omega^2\sigma\right)\right\}\boldsymbol{n}\\ &+\left\{\dot{m}\,u\omega\sigma + m\left(2u\omega\dot{\sigma} + \omega\sigma\dot{\sigma}\frac{du}{d\sigma} + u\dot{\omega}\sigma\right)\right\}\boldsymbol{t}\\ &+\boldsymbol{j}\left\{\dot{m}\,\left(c^2u^2 - \dot{\sigma}^2 - u^2\omega^2\sigma^2\right)^{1/2}\right.\\ &\left.+m\left(\frac{c^2u\dot{\sigma}\frac{du}{d\sigma} - \dot{\sigma}\ddot{\sigma} - u\omega^2\sigma^2\dot{\sigma}\frac{du}{d\sigma} - u^2\omega\dot{\omega}\sigma^2 - u^2\omega^2\sigma\dot{\sigma}}{\left(c^2u^2 - \dot{\sigma}^2 - u^2\omega^2\sigma^2\right)^{1/2}}\right)\right\}\end{aligned} \tag{3.2}$$

where in taking the derivatives of the unit vectors $\boldsymbol{n}$ and $\boldsymbol{t}$, the relationships of (2.7) and (2.8) have been inserted.
Equation (3.2) gives the reaction of the particle to changes in its Existence Momentum and, if the cause of gravitation is purely spatial, then the temporal component will be zero, so that

$$\frac{\dot{m}}{m} = -\frac{\left(c^2u\dot{\sigma}\frac{du}{d\sigma} - \dot{\sigma}\ddot{\sigma} - u\omega^2\sigma^2\dot{\sigma}\frac{du}{d\sigma} - u^2\omega\dot{\omega}\sigma^2 - u^2\omega^2\sigma\dot{\sigma}\right)}{\left(c^2u^2 - \dot{\sigma}^2 - u^2\omega^2\sigma^2\right)} \tag{3.3}$$

This naturally integrates immediately to give:-

$$\ln m = -\frac{\ln\left(c^2u^2 - \dot{\sigma}^2 - u^2\omega^2\sigma^2\right)}{2} + k \tag{3.4}$$

Initial conditions may be chosen to correspond to an apse of the spatial trajectory so that when $\dot{\sigma} = 0, m = m_0, \omega = \omega_0, u = u_0$, and $\sigma = \sigma_0$ giving

$$k = \ln\, m_0 \left(c^2 u_0^2 - u_0^2 \omega_0^2 \sigma_0^2\right)^{1/2} \tag{3.5}$$

Note that m_0 is not the rest mass but the energy mass at the apse.
Eq.(3.5) inserted into (3.4) gives

$$m = \frac{m_0 \left(c^2 u_0^2 - u_0^2 \omega_0^2 \sigma_0^2\right)^{1/2}}{\left(c^2 u^2 - \dot{\sigma}^2 - u^2 \omega^2 \sigma^2\right)^{1/2}} \tag{3.6}$$

Eq(3.6) represents the energy mass of the particle as a function of its velocity in $\mathbf{D}_1$. To eliminate the term in $\dot{\omega}$ in (3.3), use is made of the fact that gravitation is a purely radial effect with respect to the origin so that the radial normal, (t), component of (3.2) must also be zero. Therefore this gives

$$\frac{\dot{m}}{m} = -2\frac{\dot{\sigma}}{\sigma} - \frac{\dot{\sigma}}{u}\frac{du}{d\sigma} - \frac{\dot{\omega}}{\omega} \tag{3.7}$$

so that

$$-u^2 \omega \dot{\omega} \sigma^2 = u^2 \omega^2 \sigma^2 \left(\frac{\dot{m}}{m} + 2\frac{\dot{\sigma}}{\sigma} + \frac{\dot{\sigma}}{u}\frac{du}{d\sigma}\right) \tag{3.8}$$

Note that this is identical to the statement that angular momentum is constant. Substitution of (3.8) into (3.3) then gives after reduction

$$\frac{\dot{m}}{m} = -\frac{\left(c^2 u \dot{\sigma}\dfrac{du}{d\sigma} - \dot{\sigma}\ddot{\sigma} + u^2 \omega^2 \sigma \dot{\sigma}\right)}{\left(c^2 u^2 - \dot{\sigma}^2\right)} \tag{3.9}$$

so that with substitution of (3.9) into (3.2), both the radial normal, (t), and the temporal components vanish and there is left after reduction

$$\frac{d\mathbf{M}}{d\tau} = \frac{m\left(\ddot{\sigma} - u^2 \omega^2 \sigma - \dfrac{\dot{\sigma}^2}{u}\dfrac{du}{d\sigma}\right)\boldsymbol{n}}{\left(1 - \dfrac{\dot{\sigma}^2}{c^2 u^2}\right)} \tag{3.10}$$

This represents the resultant reaction of the gravitating mass to changes in its Existence Momentum.
Next the variation of the total energy of the mass as a function of its radial position from the origin will be determined for comparison with (3.10).
The total energy of matter in $\mathbf{D}_0$ was, in [1], shown to be the product of its energy mass and the square of the magnitude of its Existence Velocity. Extending this to the Domain $\mathbf{D}_1$, the total energy of the mass here is given by

$$E = mc^2 u^2 \tag{3.11}$$

Differentiating this with respect to σ gives

$$\frac{dE}{d\sigma} = \frac{dm}{d\sigma} c^2 u^2 + 2mc^2 u \frac{du}{d\sigma} \tag{3.12}$$

Converting the differential of the mass to one involving the time then gives

$$\frac{dE}{d\sigma} = \frac{\dot{m}}{\dot{\sigma}} c^2 u^2 + 2mc^2 u \frac{du}{d\sigma} \tag{3.13}$$

(3.9) may now be substituted for $\dot{m}$ to give after reduction

$$\frac{dE}{d\sigma}=m\frac{\left(c^2u\frac{du}{d\sigma}+\ddot{\sigma}-u^2\omega^2\sigma-2\frac{\dot{\sigma}^2}{u}\frac{du}{d\sigma}\right)}{\left(1-\frac{\dot{\sigma}^2}{c^2u^2}\right)} \tag{3.14}$$

This represents the resultant spatial variation of total energy with radial distance. Now comparing (3.14) with the magnitude of (3.10) it is clear that

$$\frac{dE}{d\sigma}=\frac{dM}{d\tau}+mc^2u\frac{du}{d\sigma} \tag{3.15}$$

However, for purely gravitational motion, there is no artificially applied force and therefore the total energy of such a free particle within $\mathbf{D}_1$ will be constant, i.e.

$$\frac{dE}{d\sigma}=0 \tag{3.16}$$

Insertion of this into (3.15) then gives

$$\frac{dM}{d\tau}=-mc^2u\frac{du}{d\sigma} \tag{3.17}$$

This relationship shows that the cause of gravitational motion in $\mathbf{D}_1$ is a reaction force generated within the particle proportional to it's energy mass. The term $-c^2u\frac{du}{d\sigma}$ has the dimensions of acceleration and as can be seen is solely a function of the characteristics of the Domain. For this reason this term is now defined as the Gravitational Acceleration Potential of $\mathbf{D}_1$

3.2 The Equation of Motion.

The equation of motion of the gravitating mass may now be obtained by the simple substitution of the magnitude of (3.10) into (3.17), the result being

$$\ddot{\sigma}=-c^2u\frac{du}{d\sigma}+2\frac{\dot{\sigma}^2}{u}\frac{du}{d\sigma}+u^2\omega^2\sigma \tag{3.18}$$

This is also evident from (3.14) when (3.16) is inserted. Equation (3.18) is the equation of free planar motion of a mass within $\mathbf{D}_1$. The term in $u^2\omega^2\sigma$ is the centripetal acceleration resulting from the rotational nature of the motion about the origin. The term in $2\frac{\dot{\sigma}^2}{u}\frac{du}{d\sigma}$ is an acceleration caused by the radial velocity as the particle mass moves through the varying temporal field surrounding the gravitational source. Both this and the centripetal term act in opposition to the main gravitational term, the Acceleration Potential.
The nature of the gravitational motion is clearly determined by the sign of the gradient of u, and it will be shown later that this sign is positive for a Domain identical to the Solar System.

3.3 Mass and Energy.

It was shown in [1] that in $\mathbf{D}_0$the cause of the variation of mass as a function of spatial velocity was the manner in which artificially induced kinetic energy was stored. In $\mathbf{D}_1$, for the type of motion under consideration, by virtue of (3.16) artificially induced kinetic energy does not exist and consequently the variation of mass, as exhibited by

(3.6) must have a different cause. To investigate this, (3.16) is substituted into (3.12) to give, after separation of variables

$$\frac{dm}{d\sigma} = -2\frac{m}{u}\frac{du}{d\sigma} \tag{3.19}$$

Solution of this simple equation gives

$$m = m_0\frac{u_0^2}{u^2} \tag{3.20}$$

showing that in $\mathbf{D}_1$ because u is a function of σ, energy mass is solely a function of position on the radius vector from the origin. Now substitution of (3.20) back into (3.11) then gives

$$E = m_0 c^2 u_0^2 \tag{3.21}$$

e.g. the constant value of the total energy of the gravitating mass which is seen to be that at the point taken for initial conditions.
As mentioned above, m, by virtue of the function u, is solely a function of σ. However, because m is the mass equivalent of E which, being constant for all σ, therefore indicates that in (3.20) it cannot be the amount of matter energy that is varying, but some other parameter associated with $\mathbf{D}_1$. The only other parameter involved is u and the mechanism behind the variation of mass derives from the fact that u is a measure of the temporal rate of $\mathbf{D}_1$ and if the motion of a mass involves movement along a radius vector from the origin, it therefore moves continuously through a varying temporal rate. Because the units of mass include the square of time, these, and consequently the value of mass must vary along σ according to the square of the function u.

3.4 Weight

Now (3.16) and (3.17) indicate that the gravitationally accelerated condition of the mass is its natural state of existence in $\mathbf{D}_1$. To change this state of existence, in either assisting or resisting the gravitational effect, energy must be provided. As an example of this consider the simple case in which gravitational motion of a particle mass is prevented at some distance σ_1 from the centre of the source. Thus in (3.15) putting

$$\frac{dM}{d\tau} = 0 \tag{3.22}$$

gives

$$\frac{dE}{d\sigma} = mc^2 u\frac{du}{d\sigma} \tag{3.23}$$

But because there is no motion both sides of (3.23) must be constant and it can be written

$$F_g = m_1 c^2 u_1\left(\frac{du}{d\sigma}\right)_1 \tag{3.24}$$

where F_g is the constant force applied to resist gravitation and is therefore a measure of the weight of the particle mass. Note that the weight of any particle mass is a variable dependent upon its radius vector position from the origin of the gravitational source. Therefore, if the particle were far enough away from the origin, u would become constant, its gradient zero and therefore the weight of the particle also zero. This is of course entirely in keeping with experience within the Solar System. Note also that if the sign of $\frac{du}{d\sigma}$ were negative, (3.24) shows that the weight of the particle mass in an anti-gravitational field would be negative.

4 Correlation Between the Domain $\mathbf{D}_1$ and the Solar System

To this point the analysis has been somewhat generalised because the function u, and therefore $\mathbf{D}_1$, has only been partially defined. u is the most important parameter associated with $\mathbf{D}_1$ dictating its inherent characteristics and those of existence within it. u could be specified to be any arbitrary function of σ, the resulting hypothetical domains thereby exhibiting gravitational and other characteristics of various types and degrees. To determine $\mathbf{D}_1$ such that it possesses the gravitational characteristics of ponderable matter in the Solar System, requires therefore the determination of the appropriate function for u.

4.1 Determination of the Function u.

Because gravitation is a purely radial effect, the function u can most easily, and without any loss of generality, be determined by establishing a correlation between Newton's gravitational equation for free rectilinear motion, and the appropriate approximated form of (3.18).
The former is given by

$$\frac{d^2r}{dt^2} = -\frac{\gamma\, m_g}{r^2} \tag{4.1}$$

Where r is the distance to the centre of the gravitational source, m_g is the generating mass of the source and γ is Newton's constant of proportionality. Now (4.1) is expressed in terms of the spatial and temporal axes of $\mathbf{D}_0$, Pseudo-Euclidean Space-Time. However, this relationship was empirically derived within primarily the gravitational influence of the Earth, and consequently the most accurate form of it would be obtained by its expression in terms of the spatial and temporal axes of a Domain representing the Earth's gravitational field. Equation (4.1) would then be the classical approximation to such an equation. If $\mathbf{D}_1$ is to represent this Domain then Newton's rectilinear gravitational equation expressed in the spatial and temporal axes of $\mathbf{D}_1$ would be

$$\ddot{\sigma} = -\frac{\gamma\, m_g}{\sigma^2} \tag{4.2}$$

and σ must differ from r and, over short periods, τ from t by incremental amounts not discernible in Newton's experimentation. These conditions will be proved later.
The equation of free rectilinear motion in $\mathbf{D}_1$ is obtained from (3.18) by putting ω to zero, viz.

$$\ddot{\sigma} = -c^2 u\frac{du}{d\sigma} + 2\frac{\dot{\sigma}^2}{u}\frac{du}{d\sigma} \tag{4.3}$$

Comparing (4.2) with (4.3), as the former is independent of a velocity term, the appropriate approximation of (4.3) is obtained by ignoring the term involving $\dot{\sigma}$. This merely means that in Newton's experimentation the effect of the velocity ratio $\frac{\dot{\sigma}}{cu}$, (or $\frac{dr/dt}{c}$), was too small to be observed. Thus equating the final approximation of (4.3) with (4.2) gives

$$\frac{\gamma\, m_g}{\sigma^2} = c^2 u\frac{du}{d\sigma} \tag{4.4}$$

and where there now appears on the right hand side the Gravitational Acceleration Potential of $\mathbf{D}_1$. Equation (4.4) may be integrated to give

$$-\frac{\alpha}{\sigma} = \frac{u^2}{2} + k \tag{4.5}$$

where

$$\alpha = \frac{\gamma\, m_g}{c^2} \tag{4.6}$$

is the so called gravitational radius of the source. The constant of integration is obtained by introducing the boundary condition that as $\sigma \to \infty$, $u \to 1$, i.e. as $\sigma \to \infty$, $\mathbf{D}_1 \to \mathbf{D}_0$. This gives k = -½ which when inserted into (4.5) gives the function u thus

$$u = \left(1 - \frac{2\alpha}{\sigma}\right)^{1/2} \tag{4.7}$$

From this it is easily seen that with u being positive, the gradient of u along all radius vectors from the source is also positive. The consequence is that the gravitational effect within $\mathbf{D}_1$ is to result in motion towards the origin, or gravitational source, as it does in the Solar System.
To demonstrate that $\mathbf{D}_1$ as characterised herein is truly representative of the gravitational effects of concentrated matter in the Solar System, it is necessary to provide proofs to the following three statements:

(i) That σ differs from r, and, over short periods, τ from t by incremental amounts not discernible in mechanical experimentation.

(ii) That the relationship between $\mathbf{D}_1$ and $\mathbf{D}_0$ is such that (4.1) is indeed the classical approximation of (4.2).

(iii) That free motion within $\mathbf{D}_1$ is identical to that observed in the Solar System.

The first two of these statements can be proven via establishment of the relationship between the respective polar axes, and the time, in $\mathbf{D}_1$ and $\mathbf{D}_0$.
The third proof will be demonstrated in the next section by the derivation of the equation of a central orbit in $\mathbf{D}_1$ and its comparison with that obtained from the General Theory of Relativity and with observable planetary motion in the Solar System. A rigorous solution to this equation is also presented.

4.2 Relationship Between the Polar Co-ordinate Axes of $\mathbf{D}_1$ and $\mathbf{D}_0$.

This relationship can be established by the temporal transformation of a spatial velocity in $\mathbf{D}_1$ to the Domain $\mathbf{D}_0$, and comparing the resulting co-ordinate terms with the equivalent parameters in $\mathbf{D}_0$.
Thus taking the spatial component of (2.11)

$$\bar{v} = \dot{\sigma}\boldsymbol{n} + u\omega\sigma\boldsymbol{t} \tag{4.8}$$

The equivalent expression in the co-ordinates of $\mathbf{D}_0$ is

$$\boldsymbol{v} = \frac{dr}{dt}\boldsymbol{n} + \frac{d\varphi}{dt}r\boldsymbol{t} \tag{4.9}$$

Consider the radial component of (4.8) first

$$\dot{\sigma} = \frac{d\sigma}{d\tau} \tag{4.10}$$

temporally transforming this to the Domain $\mathbf{D}_0$ using (2.16)

$$u\dot{\sigma} = \frac{d\sigma}{dt} \tag{4.11}$$

Comparing this with the radial component of (4.9), if

$$\frac{d\sigma}{dt} = \frac{dr}{dt} \tag{4.12}$$

then upon integration

$$\sigma = r + k \tag{4.13}$$

The constant of integration relates to the boundary conditions of the two Domains. The lower boundary condition is not known because $\mathbf{D}_1$ is not homogeneous in this region. However, the other boundary at which both domains are homogeneous can be utilised as follows. From (4.7) write

$$u^2\sigma^2 = \sigma^2 - 2\alpha\sigma \tag{4.14}$$

differentiate this with respect to r.

$$\frac{d}{dr}\left(u^2\sigma^2\right) = 2\frac{d\sigma}{dr}\left(\sigma - \alpha\right) \tag{4.15}$$

Inserting (4.12) and (4.13) into the right hand side of (4.15) then gives

$$\frac{d}{dr}\left(u^2\sigma^2\right) - 2r = 2\left(k - \alpha\right) \tag{4.16}$$

and this expression must conform to the boundary condition that as $r \rightarrow \infty$, $\sigma \rightarrow r$ and $u \rightarrow 1$. i.e. $\mathbf{D}_1 \rightarrow \mathbf{D}_0$. At this boundary the left-hand side of (4.16) vanishes leaving

$$k = \alpha \tag{4.17}$$

Thus in (4.12) this gives simply

$$\sigma = r + \alpha \tag{4.18}$$

as the relationship between the radial axes of $\mathbf{D}_1$ and $\mathbf{D}_0$. This expression, together with (4.6), shows that σ differs from r by an incremental amount not discernible in mechanical experimentation. i.e. proof of the spatial part of statement (i) above. It should be noted that the above process is essentially the same as that in [2] where the same result is obtained from the requirement that the two sets of co-ordinates be "harmonic". It should also be noted that (4.7) and (4.18) are only valid in homogeneous regions of $\mathbf{D}_1$ i.e. outside the generating mass of the gravitational source. This "extension" of radial distance occurs inside the gravitational source and (4.18) is the resultant effect outside the geometric dimensions of the source.

Now consider the radial normal terms in (4.8) and (4.9). Extracting and equating the angular rates, noting that the angular rate in $\mathbf{D}_1$ is already a function of time in $\mathbf{D}_0$

$$\begin{aligned} u\omega &= \frac{d\phi}{dt} \\ &= \frac{d\varphi}{dt} \end{aligned} \tag{4.19}$$

Integrating

$$\varphi = \phi \tag{4.20}$$

Where the constant of integration may be made zero by assuming a common reference radial in both domains. Thus angles in $\mathbf{D}_1$ and $\mathbf{D}_0$ are identical. This is to be expected as gravitation is a purely radial effect.

4.3 The Temporal Relationship Between $\mathbf{D}_1$ and $\mathbf{D}_0$

The temporal relationship between $\mathbf{D}_1$ and $\mathbf{D}_0$ can be established by integrating (2.19) and inserting (4.7). This gives

$$\tau = ut = t\left(\frac{r+\alpha}{r-\alpha}\right)^{1/2} \tag{4.21}$$

where the constant of integration has been put to zero by taking a common artificial temporal origin in both $\mathbf{D}_1$ and $\mathbf{D}_0$. Thus, where, as in Newton's experiments, the inequality $\sigma >> 2\frac{\gamma m_g}{c^2}$, (or $r >> \frac{\gamma m_g}{c^2}$), is valid, over short periods of time $\tau \approx t$. This constitutes proof of the temporal part of Statement (i) above.

4.4 The Classical Approximation to the Equation of Free Rectilinear Motion in $\mathbf{D}_1$

Utilising the above results it is now possible to show that Newton's gravitational equation, (4.1), is indeed the classical approximation of (4.2).
From (2.16) and (4.18)

$$\frac{d\sigma}{d\tau} = \frac{1}{u}\frac{dr}{dt} \tag{4.22}$$

so that for the left hand side of (4.2)

$$\frac{d^2\sigma}{d\tau^2} = \frac{1}{u}\frac{d}{dt}\left(\frac{1}{u}\frac{dr}{dt}\right) \tag{4.23}$$

working this out gives

$$\frac{d^2\sigma}{d\tau^2} = \frac{1}{u^2}\frac{d^2r}{dt^2} - \frac{1}{u^3}\left(\frac{dr}{dt}\right)^2\frac{du}{d\sigma} \tag{4.24}$$

Substitution for u from (4.7) and for its spatial gradient, and then for σ from (4.18) gives

$$\frac{d^2\sigma}{d\tau^2} = \left(\frac{r+\alpha}{r-\alpha}\right)\frac{d^2r}{dt^2} + \frac{\alpha}{(r-\alpha)^2}\left(\frac{dr}{dt}\right)^2 \tag{4.25}$$

Once again if $r >> \alpha$, then (4.25), for the left hand side of (4.2) approximates to

$$\frac{d^2\sigma}{d\tau^2} \approx \frac{d^2r}{dt^2} \tag{4.26}$$

For the right hand side of (4.2), substitution from (4.18) and (4.6) and taking the approximation yields

$$-\frac{\gamma m_g}{\sigma^2} \approx -\frac{\gamma m_g}{r^2} \tag{4.27}$$

Thus completing the exercise and providing proof of Statement (ii) above.

5 Planetary Orbits in $\mathbf{D}_1$ – Equation of the Orbit and its Solution.

The equation of a planetary orbit can be derived in either the axes of $\mathbf{D}_1$ or those of $\mathbf{D}_0$. In the former it is generally known as Einstein's Equation of Planetary Motion. The latter is derived in [2] from the metric of the space–time of the General Theory. The former is derived here from foregoing results as a first demonstration that gravitational motion in $\mathbf{D}_1$ is identical to that in the space-time of the General

Theory. A solution of the orbit is also obtained which when approximated for the appropriate astronomical situations shows that it also satisfactorily represents gravitational motion in the Solar System. As such this solution thereby provides the proof for Statement (iii) in Section 4 above. The equation of the orbit in the axes of $\mathbf{D}_0$ is derived in Appendix B for comparison with that in [2].

5.1 A Planetary Orbit in the Axes of $\mathbf{D}_1$ – Einstein's Equation of Planetary Motion.

Note – In this Section reference is made to results obtained in Appendix C.
To derive the equation of planetary motion in the axes of $\mathbf{D}_1$, consider first (3.7). This equation can be integrated immediately to give

$$\ln m = \ln u\omega\sigma^2 + k \tag{5.1}$$

Inserting the usual initial condition determines the constant of integration as

$$k = \ln m_0 u_0 \omega_0 \sigma_0^2 \tag{5.2}$$

so that in (5.1)

$$mu\omega\sigma^2 = m_0 u_0 \omega_0 \sigma_0^2 \tag{5.3}$$

Substituting for m from (3.6) then gives

$$\frac{\omega\sigma^2}{\left(1 - \frac{\dot{\sigma}^2}{c^2u^2} - \frac{\omega^2 r^2}{c^2}\right)^{1/2}} = \frac{\omega_0\sigma_0^2}{\left(1 - \frac{\omega_0^2\sigma_0^2}{c^2}\right)^{1/2}} \tag{5.4}$$

In line with convention, this constant is now designated as h.However, from (C7) it can also be written as

$$h = \omega'\sigma^2 \tag{5.5}$$

The equation of the orbit can now be derived in the usual manner as follows. Put

$$\sigma = \frac{1}{\mu} \tag{5.6}$$

and from (5.5) and (5.6) compute the second order differential term in (C9) as

$$\frac{d^2\sigma}{d\tau_p^2} = -\mu^2 h^2 \frac{d^2\mu}{d\phi^2} \tag{5.7}$$

Substitution of this, together with (5.5) and (5.6) into (C9) then yields the desired equation of the orbit thus

$$\frac{d^2\mu}{d\phi^2} + \mu = \frac{\alpha c^2}{h^2} + 3\alpha\mu^2 \tag{5.8}$$

5.2 Solution of the Orbital Equation in $\mathbf{D}_1$

In the literature the equation of the trajectory of a planetary orbit has been obtained from an approximate solution of the orbital equation for two particular cases. Firstly, for closed orbits, it has been shown to approximate a precessing ellipse, viz. [2] pp 199, which constructs an approximate solution of (B14), (the equation of the orbit expressed in the axes of $\mathbf{D}_0$), for a truly elliptical orbit. A similar result is obtained in [3], pp247 Example 102, where an approximate solution of (5.8) is obtained. This

solution however, applies only to a circular orbit. Secondly, an approximate solution of (B14) has been obtained for an open orbit in the extreme form of a light ray passing close to the geometrical radius of a gravitational source, viz. [2] pp 202, and has thus been shown to be a precessing hyperbola.
In this Section, an exact solution is obtained for the planetary orbit in $\mathbf{D}_1$, (5.8), by assuming it to be a precessing conic section and using, essentially, the method of Frobenús. This solution is then reduced to the above approximate forms via a process of logical simplification.
Note – In this Section reference is made to results obtained in Appendices B and C.

5.2.1 Equation of the Spatial Trajectory – The Basic Curve.

The equation of the basic curve of the trajectory will be derived from the first integral of (5.8) which is á priori obtained from (B8) by computing $d\sigma/d\tau_p$ from the first order term in (C2), then substitution of (5.5), (5.6) and (B16) to give

$$\left(\frac{d\mu}{d\phi}\right)^2 = \frac{c^2\varepsilon^2}{h^2} - \frac{cu^2}{h^2} - u^2\mu^2 \tag{5.9}$$

Substitution of (4.7) then gives the required relationship as

$$\left(\frac{d\mu}{d\phi}\right)^2 = \frac{c^2}{h^2}\left(\varepsilon^2 - 1\right) + \frac{2\alpha\, c^2}{h^2}\mu - \mu^2 + 2\alpha\mu^3 \tag{5.10}$$

the first integral of (5.8).
If the integral of (5.10) is assumed to be a precessing conic section then it will take the form

$$\mu = \frac{1}{L}\left\{1 + e\cos\left(\phi - \Omega\right)\right\} \tag{5.11}$$

where

- L is the semi latus rectum
- e is the eccentricity of the basic curve
- ϕ is the angle of the focal point radius vector to the major axis
- Ω is the angle of precession of an apse of the trajectory.

Differentiating (5.11) with respect to ϕ yields

$$\frac{d\mu}{d\phi} = -\frac{e}{L}\sin\left(\phi - \Omega\right)\left(1 - \frac{d\Omega}{d\phi}\right) \tag{5.12}$$

but from (5.11)

$$\sin\left(\phi - \Omega\right) = \left\{1 - \left(\frac{\mu L - 1}{e}\right)^2\right\}^{1/2} \tag{5.13}$$

Thus (5.12) and (5.13) give, after expansion

$$\left(\frac{d\mu}{d\phi}\right)^2 = \left\{\frac{\left(e^2 - 1\right)}{L^2} + \frac{2\mu}{L} - \mu^2\right\}\left(1 - \frac{d\Omega}{d\phi}\right)^2 \tag{5.14}$$

Comparing (5.14) with (5.10), to obtain the term in μ^3 with the correct coefficient, it is necessary to put

$$\left(1 - \frac{d\Omega}{d\phi}\right)^2 = \left(b - 2\alpha\mu\right) \tag{5.15}$$

where b is some constant.

Inserting (5.15) into (5.14) and expanding gives

$$\left(\frac{d\mu}{d\phi}\right)^2 = \frac{\left(e^2-1\right)}{L^2}b + \frac{2}{L}\left\{b - \frac{\alpha}{L}\left(e^2-1\right)\right\}\mu - \left(b + \frac{4\alpha}{L}\right)\mu^2 + 2\alpha\mu^3 \tag{5.16}$$

Again, comparing (5.16) with (5.10), for the coefficient of μ^2 to be -1, it is necessary to put

$$b = 1 - \frac{4\alpha}{L} \tag{5.17}$$

so that in (5.16) this gives after reduction

$$\left(\frac{d\mu}{d\phi}\right)^2 = \frac{\left(e^2-1\right)}{L^2}\left(1 - \frac{4\alpha}{L}\right) + \frac{2}{L}\left\{1 - \frac{\alpha}{L}\left(3+e^2\right)\right\}\mu - \mu^2 + 2\alpha\mu^3 \tag{5.18}$$

Now, equating the coefficient of μ in (5.18) with that in (5.10) then gives

$$h^2 = \frac{\alpha c^2 L}{1 - \frac{\alpha}{L}\left(3+e^2\right)} \tag{5.19}$$

L can be determined from initial conditions applied to (5.11) to be

$$L = \frac{1+e}{\mu_0} \tag{5.20}$$

Inserting (5.20) and (5.4) for h into (5.19) then gives, after solving for ω_0^2

$$\omega_0^2 = \frac{\alpha c^2\left(1+e\right)^2\mu_0^3}{\left\{1 - 2\alpha\mu_0 + \left(1+2\alpha\mu_0\right)e\right\}} \tag{5.21}$$

Now, for the basic orbit to be circular, i.e. $e=0$, (5.21) reduces to

$$\omega_0^2 = \frac{\alpha c^2\mu_0^3}{1 - 2\alpha\mu_0} \tag{5.22}$$

For values of ω_0 below this, the basic orbit will be degenerative, and for values above elliptical.
For the basic orbit to be parabolic, i.e. $e=1$, (5.21) reduces to

$$\omega_0^2 = 2\alpha c^2\mu_0^3 \tag{5.23}$$

For values of $\omega_0$below this the basic orbit will be elliptical and for values above hyperbolic.
Solving (5.21) for e gives

$$e = \frac{1}{2\alpha\mu_0}\left[\frac{\omega_0^2}{c^2\mu_0^2}\left(1+2\alpha\mu_0\right) - 2\alpha\mu_0 \pm \frac{\omega_0}{c\mu_0}\left\{\frac{\omega_0^2}{c^2\mu_0^2}\left(1+2\alpha\mu_0\right)^2 - 16\alpha^2\mu_0^2\right\}^{1/2}\right] \tag{5.24}$$

which can also be obtained by equating the constant terms in (5.10) and (5.18).

Now, for the case in which $\omega_0$is equal or very close to its terminal value in $\mathbf{D}_1$, its upper boundary, i.e. $\omega_0 = c\mu_0$,(5.24) reduces to

$$e = \frac{1}{2\alpha\mu_0}\left\{1 \pm \left(1 + 4\alpha\mu_0 - 12\alpha^2\mu_0^2\right)^{1/2}\right\} \tag{5.25}$$

and it is noted for future reference that if $4\alpha\mu_0 << 1$, (5.25) may be approximated by

$$e \approx \frac{1}{\alpha\mu_0} \tag{5.26}$$

There appears to be a second root under this criteria, i.e. at $e = 0$.However, if the terminal value of ω_0 is entered into (5.22), it reduces to $3\alpha\mu_0 = 1$,which contradicts the inequality. This second root may therefore be discounted for astronomical situations. Thus, the basic curve of the trajectory is now seen to be given by the combination of the three equations; (5.11), (5.20) and (5.24). The full solution to (5.10) is completed by the determination of the function Ω.

5.2.2 *Precession of the Perihelion of the Basic Curve.*

Determination of Ω is effected by substituting (5.11), (5.17) and (5.20) into (5.15) to give

$$\left(1 - \frac{d\Omega}{d\phi}\right)^2 = 1 - \frac{6\alpha\mu_0}{1+e} - \frac{2\alpha\mu_0 e}{1+e}\cos(\phi - \Omega) \tag{5.27}$$

Putting

$$\phi - \Omega = \chi \tag{5.28}$$

reduces (5.27) to the following elliptic integral

$$d\phi = \left(1 - \frac{6\alpha\mu_0}{1+e} - \frac{2\alpha\mu_0 e}{1+e}\cos\chi\right)^{-1/2} d\chi \tag{5.29}$$

the solution of which is

$$\begin{aligned}\phi = \left(1 - \tfrac{6\alpha\mu_0}{1+e}\right)^{-1/2} \Bigg\{ &\left(1 + \frac{3\alpha^2\mu_0^2 e^2}{(1+e - 6\alpha\mu_0)}\right)\chi \\ &- \frac{\alpha\mu_0 e \sin\chi}{(1+e-6\alpha\mu_0)} + \frac{3\alpha^2\mu_0^2 e^2 \sin 2\chi}{2\,(1+e-6\alpha\mu_0)} - \ldots \Bigg\}\end{aligned} \tag{5.30}$$

the constant of integration being zero.
This relationship, together with (5.28) then permits the determination of Ω for any value of ϕ, for any condition. Thus (5.30), together with (5.11), (5.20) and (5.24), albeit somewhat cumbersome, constitutes the exact solution of (5.10).

5.2.3 *Comparison with Existing Approximate Solutions.*

To compare the above results with the approximate solutions of the General Theory mentioned earlier, it is only necessary to simplify (5.30) for known astronomical conditions.
Firstly, terms in μ_0^2 may be considered negligible in comparison with unity, (5.30) then becomes

$$\phi \approx \left(1 - \frac{6\alpha\mu_0}{1+e}\right)^{-1/2} \left\{\phi - \Omega - \frac{\alpha\mu_0 e \sin(\phi - \Omega)}{(1+e-6\alpha\mu_0)}\right\} \tag{5.31}$$

in which (5.28) has been re-inserted.
For astronomical conditions, Ω will be very small compared to ϕ so that (5.31) may be further simplified to

$$\phi \approx \left(1 - \frac{6\alpha\mu_0}{1+e}\right)^{-1/2} \left\{\phi - \Omega + \frac{\alpha\mu_0 e}{(1+e-6\alpha\mu_0)}\left(\Omega\cos\phi + \sin\phi\right)\right\} \tag{5.32}$$

This is solvable for Ω giving

$$\Omega \approx \frac{\left\{1 - \left(1 - \frac{6\alpha\mu_0}{1+e}\right)^{1/2}\right\}\phi + \frac{\alpha\mu_0 e\sin\phi}{(1+e-6\alpha\mu_0)}}{\left(1 - \frac{\alpha\mu_0 e\cos\phi}{(1+e-6\alpha\mu_0)}\right)} \tag{5.33}$$

Now, if the astronomical situation is such that the additional inequality $\alpha\mu_0 << 1$ is valid, (5.33) finally reduces to

$$\Omega \approx \frac{3\alpha\mu_0}{1+e}\phi + \frac{\alpha\mu_0 e\sin\phi}{1+e} \tag{5.34}$$

Consider first, an orbit with an elliptical basic curve. When ϕ=2π, (5.34) becomes

$$\Omega \approx \frac{6\pi\alpha\mu_0}{1+e} \tag{5.35}$$

in agreement with the solution of [2], pp199, Eq(58.43).
For a circular orbit, $e = 0$ which gives in (5.34)

$$\Omega \approx 3\alpha\mu_0\phi \tag{5.36}$$

However, μ_0 may be approximated for a circular orbit from (5.19) and (5.20) to be

$$\mu_0 \approx \frac{\alpha c^2}{h^2} \tag{5.37}$$

Substitution of this into (5.36) then yields

$$\Omega \approx \frac{3\alpha^2 c^2}{h^2}\phi \tag{5.38}$$

which agrees with the approximate solution of [3], pp 247, Example 102, and represents a maximum value of Ω.
For the special case in which $\omega_0 = c\mu_0$, its upper boundary, and the geometric radius of the gravitational source is much greater than its gravitational radius, then (5.26) may be used for the eccentricity of the basic trajectory. Thus inserting (5.26) into (5.34) gives

$$\Omega \approx \frac{3\alpha^2\mu_0^2}{1+\alpha\mu_0}\phi + \frac{\alpha\mu_0\sin\phi}{1+\alpha\mu_0} \tag{5.39}$$

which may immediately be further approximated to

$$\Omega \approx \alpha\mu_0\sin\phi \tag{5.40}$$

This represents the rotation of the perihelion of the hyperbolic curve and constitutes a minimum value of Ω. To determine the total angle of deflection of the trajectory, (5.20), (5.26) and (5.40) are now substituted into (5.11) to give

$$\mu \approx \frac{\mu_0}{1+\alpha\mu_0}\left\{\alpha\mu_0 + \cos\left(\phi - \alpha\mu_0\sin\phi\right)\right\} \tag{5.41}$$

expanding the cosine term and taking the usual approximation yields after reduction

$$\mu \approx \mu_0 \left(2\alpha\mu_0 + \cos\phi\right) \tag{5.42}$$

The total angle of deflection may now be approximated from the two boundary conditions of the orbit where $\mu = 0$.At these boundaries if $\phi = \pm\left(\frac{\pi}{2} + \delta\right)$, then applying these conditions to (5.42) gives

$$\delta \approx 2\alpha\mu_0 \tag{5.43}$$

and the total angle of deflection of the trajectory is then

$$2\delta \approx 4\alpha\mu_0 \tag{5.44}$$

in agreement with the approximate solution in [2], pp202, Eq[59.18], for the "bending of light rays" in close proximity to a gravitational source.
The results of this Section provide the final proof for Statement (iii) in Section 4.

6 Concluding Remarks

The conventional mathematical approach adopted here has, for the gravitational effects studied, produced results that are in agreement with those obtained from the General Theory of Relativity, and as observed for astronomical motion within the Solar System. However, the concepts from which this approach stems, e.g. that all matter must possess an Existence Velocity within a linear Relativistic Domain, as herein defined, differ from the concepts upon which the General Theory is based. That difference is in the manner in which the "gravitational field" of the source causes motion. In the General Theory this is defined as due to a "curvature" of the space-time continuum, proportional to, and in the direction of, the gravitational source. The "world line" of any gravitating mass is then said to be curved in the direction of the source.

In the development presented here, the space-time continuum of the Relativistic Domain $\mathbf{D}_1$ is defined to be spatially and temporally linear, and gravitationally induced motion has been shown to be caused by an Acceleration Potential, generated by the gravitational source, proportional to its mass and inversely proportional to the distance from it. The effect of this Potential has in turn been shown to be augmented by the result of temporal dilatation produced by the source.
Because of this fundamental difference in which gravitational motion is caused, the concepts presented in this paper cannot be considered as merely a different mathematical formulation of gravitation, but should be considered as an alternative to that of the General Theory.

In both concepts the continuum of Pseudo-Euclidean Space-Time, is required to possess characteristics such that it interacts with matter energy to produce a new continuum, which in turn causes gravitationally induced motion. Also, both concepts incorporate temporal dilatation plus the small radial extension of distance from the source. Both of these latter effects are created within the body of the gravitational source and then extend beyond its geometric confines. In the General Theory temporal dilatation is treated as a consequence of gravitation while in the presentation here it is shown to contribute to the cause of it. Accordingly, in view of the great similarity of results in the two concepts, it is believed that the purported cause of gravitationally induced motion in the General Theory, the curvature of space-time, is a mis-interpretation, and this curvature is nothing more than the curvature of the trajectory of the gravitating mass, rather than of the space-time continuum in which it moves. If this is so, the consequence is that the continuum proper of the General Theory must therefore be identical to that of $\mathbf{D}_1$. Some evidence for this is shown in Appendix D. Also, the derivation of Einstein's equation of planetary motion from the characteristics of $\mathbf{D}_1$ further supports this opinion.

The primary cause of gravitational motion in the Domain $\mathbf{D}_1$ is its Acceleration Potential. This Potential is in turn augmented by the spatially dependent temporal dilatation shown to exist in that Domain, and represented by the parameter u. The generation of these effects within a gravitational source in $\mathbf{D}_1$ is therefore central to this theory, and a mechanism for it will be presented in a next paper.

The successful application of the concepts of Relativistic Domains in both a Pseudo-Euclidean Space-Time, ($\mathbf{D}_0$, see [1]), and a Gravitational one ($\mathbf{D}_1$), as well as representing a unification of mathematical analysis within them, has also established a unique link between them such that the variation of only one parameter, the temporal rate u, is sufficient to transform one into the other. In fact this can be completely generalised with the result that both the Pseudo-Euclidean and Gravitational space-times are merely particular cases of a potentially infinite number of hypothetical Relativistic Domains in which the parameter u can be of any form. Pseudo-Euclidean Space-Time would perhaps retain its special character insofar as its temporal rate possessed the unique value of unity. The Domain $\mathbf{D}_1$is of course equally if not more important because it describes the space time continuum of the gravitational effects within Solar System and beyond.

Throughout the text there are a number of results that can be taken further. Of particular interest are the questions of inertial mass, naturally generated kinetic energy and effects inside the geometric constraints of the gravitational source itself. The same applies when given a set of special conditions outside the source. Most particular in this latter respect is the situation, possibly hypothetical, where the geometrical radius of a gravitational source is of the same order of magnitude as twice the gravitational radius. At the point of equivalence the temporal rate of $\mathbf{D}_1$ becomes zero, i.e. time stops.

Finally, it should be remembered that, for a single isolated gravitational source, despite the agreement of the results in this paper with those of the General Theory, and with observed planetary motion in the Solar System, all such results, strictly from the point of view of mathematical rigour, are approximations. This is so because in all cases the analysis assumes that the gravitational source is stationary, i.e. not affected by the gravitational influence of the gravitating mass. No matter how small the latter, there will always be an effect on the larger mass. Therefore the results here and in the General Theory are only approximately correct in the case where the gravitational source is much larger than the gravitating mass. Where sizes are comparable it is necessary to take account of the mutual gravitational attraction, which effectively results in a new Relativistic Domain, $\mathbf{D}_2$.

The appendices to the main text provide additional evidence that gravitational motion within the Relativistic Domain $\mathbf{D}_1$ is identical to that postulated in the General Theory.

APPENDIX A

Transformation of the Equation of Free Planar Motion to the Axes of $\mathbf{D}_0$

As a further example of the relationship between $\mathbf{D}_1$ and $\mathbf{D}_0$, the equation of planar motion in $\mathbf{D}_1$ (3.18) is, in this appendix, transformed to the axes of $\mathbf{D}_o$.
Transformation of the term $\ddot{\sigma}$ in (3.18) is given in terms of u by (4.24). Transformation of the other terms is as follows.
From (2.19) and (4.18)

$$u^2\omega^2\sigma = \left(\frac{d\phi}{dt}\right)^2 (r+\alpha) \tag{A.1}$$

and from (2.19), (4.12) and (4.18)

$$\frac{2\dot{\sigma}^2}{u}\frac{du}{d\sigma} = \frac{2}{u^3}\left(\frac{dr}{dt}\right)^2 \frac{du}{dr} \tag{A.2}$$

and finally from (4.12)

$$-c^2u\frac{du}{d\sigma} = -c^2u\frac{du}{dr} \tag{A.3}$$

The full transformation of (3.18) in terms of u then becomes, from (4.24), (A.1), (A.2) and (A.3)

$$\frac{d^2r}{dt^2} = -c^2u^3\frac{du}{dr} + \frac{3}{u}\left(\frac{dr}{dt}\right)^2\frac{du}{dr} + u^2\left(\frac{d\phi}{dt}\right)^2(r+\alpha) \tag{A.4}$$

Substitution of (4.7) and (4.18) for u and σ, then expands this to the final result

$$\frac{d^2r}{dt^2} = -\alpha c^2\frac{(r-\alpha)}{(r+\alpha)^3} + \frac{3\alpha}{r^2-\alpha^2}\left(\frac{dr}{dt}\right)^2 + (r-\alpha)\left(\frac{d\phi}{dt}\right)^2 \tag{A.5}$$

APPENDIX B

Transformation of the Equation of the Orbit to the Axes of $\mathbf{D}_0$

The equation of a planetary orbit in the axes of Pseudo-Euclidean Space-Time has, in [2] been derived, in the form of a first order equation, from a Lagrangian analysis of the metric of the General Theory. To obtain that form here, the simplest process is to obtain the first integral of (C.8) from which the desired relationship can be obtained directly. The easiest manner to obtain the first integral of (C.8) is firstly, via a re-arrangement of (5.4), thus

$$\dot{\sigma}^2 = -u^2\omega^2\sigma^2 + c^2u^2\left\{1 - \frac{\omega^2\sigma^4}{\omega_0^2\sigma_0^4}\left(1 - \frac{\omega_0^2\sigma_0^2}{c^2}\right)\right\} \tag{B.1}$$

From (5.3) note that

$$m = \frac{m_0u_0\omega_0\sigma_0^2}{u\omega\sigma^2} \tag{B.2}$$

which with (3.20) gives

$$\frac{u_0}{u} = \frac{\omega_0\sigma_0^2}{\omega\sigma^2} \tag{B.3}$$

Inserting this into (B.1) then gives

$$\dot{\sigma}^2 = -u^2\omega^2\sigma^2 + c^2u^2\left\{1 - \frac{u^2}{u_0^2}\left(1 - \frac{\omega_0^2\sigma_0^2}{c^2}\right)\right\} \tag{B.4}$$

which incidentally can be shown to be the first integral of (3.18), the equation of planar motion in $\mathbf{D}_1$.
Now, from (2.15)

$$\frac{d\sigma}{d\tau_p} = \dot{\sigma}\frac{d\tau}{d\tau_p} = \frac{\dot{\sigma}}{\left(1 - \frac{\dot{\sigma}^2}{c^2u^2} - \frac{\omega^2\sigma^2}{c^2}\right)^{1/2}} \tag{B.5}$$

so that substitution for $\dot{\sigma}$ from (B.4) yields

$$\left(\frac{d\sigma}{d\tau_p}\right)^2 = \frac{c^2u_0^2}{\left(1 - \frac{\omega_0^2\sigma_0^2}{c^2}\right)} - c^2u^2 - \frac{u_0^2\omega^2\sigma^2}{\left(1 - \frac{\omega_0^2\sigma_0^2}{c^2}\right)} \tag{B.6}$$

Also from (C.7) and (B.4)

$$\omega^2 = \omega'^2\frac{u^2}{u_0^2}\left(1 - \frac{\omega_0^2\sigma_0^2}{c^2}\right) \tag{B.7}$$

which when inserted into (B6) yields

$$\left(\frac{d\sigma}{d\tau_p}\right)^2 = \frac{c^2u_0^2}{\left(1 - \frac{\omega_0^2\sigma_0^2}{c^2}\right)} - c^2u^2 - u^2\omega'^2\sigma^2 \tag{B.8}$$

as the first integral of (C.8).

Transformation to the Axes of D_0 and Derivation of the Equation of the Orbit.

Transformation of (B.8) to the axes of $\mathbf{D}_0$ via (4.7), (4.12) and (4.13) gives

$$\left(\frac{dr}{d\tau_p}\right)^2 = \frac{c^2\left(\dfrac{r_0-\alpha}{r_0+\alpha}\right)}{1-\dfrac{\omega_0^2}{c^2}\left(r_0+\alpha\right)^2} - c^2\left(\frac{r-\alpha}{r+\alpha}\right) - \left(\frac{d\phi}{d\tau_p}\right)^2\left(r^2-\alpha^2\right) \tag{B.9}$$

For simplicity write this as

$$\left(\frac{dr}{d\tau_p}\right)^2 = c^2\varepsilon^2 - c^2\left(\frac{r-\alpha}{r+\alpha}\right) - h^2\frac{(r-\alpha)}{(r+\alpha)^3} \tag{B.10}$$

where (5.5) has also been inserted
The equation of the orbit, (expressed in the axes $\mathbf{D}_0$), can now be derived in the conventional manner as follows. Put

$$r = \frac{1}{\varsigma} \tag{B.11}$$

so that

$$\frac{dr}{d\tau_p} = -\frac{d\varsigma}{d\phi}\frac{h}{(1+\alpha\varsigma)^2} \tag{B.12}$$

Inserting this and (B.11) into (B.10) yields

$$\left(\frac{d\varsigma}{d\phi}\right)^2 = \frac{c^2\varepsilon^2}{h^2}(1+\alpha\varsigma)^4 - \frac{c^2}{h^2}(1-\alpha\varsigma)(1+\alpha\varsigma)^3 - \varsigma^2(1-\alpha\varsigma)^2 \tag{B.13}$$

Expanding, this finally reduces to the desired expression, thus

$$\begin{aligned}\left(\frac{d\varsigma}{d\phi}\right)^2 &= \frac{c^2}{h^2}\left(\varepsilon^2-1\right) + \frac{2\alpha c^2}{h^2}\left(2\varepsilon^2-1\right)\varsigma + \left(\frac{6\alpha^2c^2\varepsilon^2}{h^2}-1\right)\varsigma^2 \\ &+\frac{2\alpha^3c^2}{h^2}\left(2\varepsilon^2+1\right)\varsigma^3 + \alpha^2\left\{1+\frac{\alpha^2c^2}{h^2}\left(\varepsilon^2+1\right)\right\}\varsigma^4\end{aligned} \tag{B.14}$$

as derived in [2], pp 198, Eq[58.35].
Finally, in (B.10) the simplifying identity

$$\varepsilon = \frac{\left(\dfrac{r_0-\alpha}{r_0+\alpha}\right)^{1/2}}{\left\{1-\dfrac{\omega_0^2}{c^2}\left(r_0+\alpha\right)^2\right\}^{1/2}} \tag{B.15}$$

was inserted. To shown that this is identical to the same parameter in [2], pp197, Eq(58.26), insert (4.7) and (4.18) thus

$$\varepsilon = \frac{u_0}{\left(1-\dfrac{\omega_0^2\sigma_0^2}{c^2}\right)^{1/2}} \tag{B.16}$$

which from (3.6) and (3.20) becomes

$$\varepsilon = \frac{cu^2}{\left(c^2u^2-\dot{\sigma}^2-u^2\omega^2\sigma^2\right)^{1/2}} \tag{B.17}$$

and which with (2.15) and (2.19) then gives

$$\varepsilon = u\frac{d\tau}{d\tau_p} = u^2\frac{dt}{d\tau_p} \tag{B.18}$$

so that insertion of (4.7) again into this finally gives

$$\varepsilon = \left(\frac{r-\alpha}{r+\alpha}\right)\frac{dt}{d\tau_p} \tag{B.19}$$

As derived in [2].
Also from (5.5) it can be seen that the constant h in this paper is identical to the parameter μ in [2], pp197, Eq(58.27). These results provide additional proof that a central orbit in $\mathbf{D}_1$ is identical to that in the General Theory.

APPENDIX C

Determination of the Equation of Free Planar Motion as a Function of the Proper Time of the Gravitating Mass.

It is first noted for future reference that substitution for $\frac{\dot{m}}{m}$, derived from (3.19), into (3.8) gives

$$\dot{\omega} = -\omega\left(2\frac{\dot{\sigma}}{\sigma} - \frac{\dot{\sigma}}{u}\frac{du}{d\sigma}\right) \tag{C.1}$$

First the second order variation of radial position with respect to the proper time of the gravitating mass is computed thus

$$\frac{d^2\sigma}{d\tau_p^2} = \frac{d\tau}{d\tau_p}\frac{d}{d\tau}\left(\frac{d\tau}{d\tau_p}\frac{d\sigma}{d\tau}\right) \tag{C.2}$$

which from (2.15) becomes

$$\frac{d^2\sigma}{d\tau_p^2} = \left(1 - \frac{\dot{\sigma}^2}{c^2u^2} - \frac{\omega^2\sigma^2}{c^2}\right)^{-1/2}\frac{d}{d\tau}\left\{\dot{\sigma}\left(1 - \frac{\dot{\sigma}^2}{c^2u^2} - \frac{\omega^2\sigma^2}{c^2}\right)^{-1/2}\right\} \tag{C.3}$$

working this out yields

$$\frac{d^2\sigma}{d\tau_p^2} = \frac{\ddot{\sigma}}{\left(1 - \frac{\dot{\sigma}^2}{c^2u^2} - \frac{\omega^2\sigma^2}{c^2}\right)} + \frac{\dot{\sigma}\left(\frac{\dot{\sigma}\ddot{\sigma}}{c^2u^2} - \frac{\dot{\sigma}^3}{c^2u^3}\frac{du}{d\sigma} + \frac{\omega\dot{\omega}\sigma^2}{c^2} + \frac{\omega^2\sigma\dot{\sigma}}{c^2}\right)}{\left(1 - \frac{\dot{\sigma}^2}{c^2u^2} - \frac{\omega^2\sigma^2}{c^2}\right)^2} \tag{C.4}$$

Substitution for $\dot{\omega}$ from (C.1) then gives after reduction

$$\frac{d^2\sigma}{d\tau_p^2} = \frac{\ddot{\sigma}\left(1 - \frac{\omega^2\sigma^2}{c^2}\right) - \frac{\omega^2\sigma\dot{\sigma}^2}{c^2} - \frac{\dot{\sigma}^4}{c^2u^3}\frac{du}{d\sigma} + \frac{\omega^2\sigma^2\dot{\sigma}^2}{c^2u}\frac{du}{d\sigma}}{\left(1 - \frac{\dot{\sigma}^2}{c^2u^2} - \frac{\omega^2\sigma^2}{c^2}\right)^2} \tag{C.5}$$

Now substitution for $\ddot{\sigma}$ from (3.18) gives

$$\frac{d^2\sigma}{d\tau_p^2} = -c^2u\frac{du}{d\sigma} + \frac{u^2\omega^2\sigma - u\omega^2\sigma^2\frac{du}{d\sigma}}{\left(1 - \frac{\dot{\sigma}^2}{c^2u^2} - \frac{\omega^2\sigma^2}{c^2}\right)} \tag{C.6}$$

But

$$\omega = \frac{d\phi}{d\tau} = \frac{d\phi}{d\tau_p}\frac{d\tau_p}{d\tau} = \omega'\left(1 - \frac{\dot{\sigma}^2}{c^2u^2} - \frac{\omega^2\sigma^2}{c^2}\right)^{1/2} \tag{C.7}$$

and insertion of this into (C.6) finally gives

$$\frac{d^2\sigma}{d\tau_p^2} = -c^2u\frac{du}{d\sigma} - u\omega'^2\sigma^2\frac{du}{d\sigma} + u^2\omega'^2\sigma \tag{C.8}$$

which is the required relationship, expressed in the axes of $\mathbf{D}_1$and the parameter u.Substitution for u,(from (4.7)) and its spatial gradient, reduces (C.8) to

$$\frac{d^2\sigma}{d\tau_p^2} = -\frac{\alpha c^2}{\sigma^2} + (\sigma - 3\alpha)\,\omega'^2 \tag{C.9}$$

being the equation of motion as a function of the proper time expressed fully in the axes of $\mathbf{D}_{1.}$

Note that because the time dilatation effect is embodied in the proper time τ_p, the reactive acceleration term due to this in (3.18), is not present in (C.9).

APPENDIX D

The Red Shift of Atomic Spectra in $\mathbf{D}_1$.

This is the third classic test to which the General Theory was subjected to verify its applicability within the Solar System. It is therefore necessary that the existence of atomic spectra within $\mathbf{D}_1$ meet the same criteria.
Note that in this Appendix the value of Spatial Terminal Velocity is used for the velocity of electromagnetic radiation in a vacuum. This has been done solely to enable comparison of the results of this Appendix with similar effects in the General Theory. It may not be strictly correct however, because such radiation must possess a mass, however small, by virtue of Einstein's universal energy-mass relationship, and it is not possible, within a finite time to accelerate any mass to the Spatial Terminal Velocity of a Relativistic Domain. Accordingly, it should also be noted that the Spatial Terminal Velocity of $\mathbf{D}_1$ is by virtue of the function u,a variable dependent upon σ and so therefore is the velocity of electromagnetic radiation as defined in this Appendix.
Accordingly, the wavelength of an atomic spectra at the point of emission in $\mathbf{D}_1$, the surface of a gravitational source, a distance of σ_1 from its centre, is given by

$$\lambda_1 = \frac{cu_1}{f_1} \tag{D.1}$$

where f_1 is the frequency of the spectra, and cu_1 its velocity of propagation at the point of emission.
The solution of the rectilinear version of (3.18), (with $\omega = 0$), for an initial condition of $\dot{\sigma}_0 = cu_1$ is

$$\dot{\sigma} = cu \tag{D.2}$$

so that after travelling directly away from the source to a point of observation, a distance of σ_2 from the centre of the source, the wavelength of the spectra will be

$$\lambda''_1 = \frac{cu_2}{f''_1} \tag{D.3}$$

Another spectra of an identical atom emitted at the point of observation will possess a wavelength of

$$\lambda_2 = \frac{cu_2}{f_2} \tag{D.4}$$

so that from (D.3) and (D.4)

$$\frac{\lambda''_1}{\lambda_2} = \frac{f_2}{f''_1} \tag{D.5}$$

f''_1 is the frequency of the first spectra after travelling to the point of observation and is given by

$$f''_1 = \frac{dn_1}{d\tau_2} = \frac{dn_1}{d\tau_1}\frac{d\tau_1}{d\tau_2} = f_1\frac{u_1}{u_2} \tag{D.6}$$

where n_1 is an integral number of cycles and $d\tau_1$ and $d\tau_2$ are elements of time at the points of emission and observation respectively. u_1 and u_2 are the temporal rates at these locations. Therefore

$$\frac{\lambda''_1}{\lambda_2} = \frac{f_2 u_2}{f_1 u_1} = \frac{E_2 u_2}{E_1 u_1} \tag{D.7}$$

where E_1 and E_2 are the energies imparted to the two respective waves by the process of emission. Because this process is an internal function of the atom concerned, the energy of emission is independent of the location within the Domain in which it occurs. Therefore

$$E_1 = E_2 \tag{D.8}$$

and so

$$\lambda_1'' = \lambda_2 \frac{u_2}{u_1} \tag{D.9}$$

Because $u_2 > u_1$, λ_1''exhibits an apparent red shift compared to λ_2. Also note from (D.7) and (D.8) it is clear that $f_1 = f_2$.
If the point of observation is sufficiently far from the point of emission, it may, (as in the literature), be approximated to free space, i.e. $u_2 \to 1$, as in $\mathbf{D}_0$ and then

$$\lambda_1'' \approx \frac{\lambda_2}{u_1} \tag{D.10}$$

which after insertion of (4.7) may be further approximated to

$$\lambda_1'' \approx \lambda_2 \left(1 + \frac{\alpha}{\sigma_1}\right) \tag{D.11}$$

This is effectively the result most often quoted in the literature, [4], [5].
It should be noted that a comparison of the wavelength of the first wave upon reaching the point of observation with its wavelength at the point of emission produces the result

$$\lambda_1'' = \lambda_1 \left(\frac{u_2}{u_1}\right)^2 \tag{D.12}$$

showing that the true red shift of the travelling wave is greater than when simply compared to a wave emitted at the point of observation. Substitution of (D.12) into (D.9) then gives

$$\lambda_2 = \lambda_1 \frac{u_2}{u_1} \tag{D.13}$$

as would be expected.
The mechanism behind the shift is that as the wave moves away from the source, it continuously moves through an increasing temporal rate, which causes its frequency to decrease. This produces a corresponding increase in spectral wavelength.
Note that from (D1), if the geometrical radius of the gravitational source is equal to, (or less than), twice its gravitational radius, the propagation velocity of emission is zero. Hence electromagnetic radiation from such a physical body is impossible. This indicates that "Black Holes" are at least mathematically permissible within the Domain $\mathbf{D}_1$, as they are in the General Theory. However, it will be shown in a future paper that there are other constraints which prohibit the formation of Black Holes within the Relativistic Domain $\mathbf{D}_1$.

APPENDIX E

Derivation of the Metric of General Relativity from the Characteristics of Existence in $\mathbf{D}_1$

To establish the relationship between the metric of the General Theory and the characteristics of existence in $\mathbf{D}_1$ it is necessary to extend (2.15) into the second spatial plane thus

$$\frac{d\tau_p}{d\tau} = \left\{1 - \frac{\dot{\sigma}^2}{c^2u^2} - \left(\frac{d\phi}{d\tau}\right)^2\frac{\sigma^2}{c^2} - \left(\frac{d\beta}{d\tau}\right)^2\frac{\sigma^2}{c^2}\sin^2\phi\right\}^{1/2} \tag{E.1}$$

where β is an angle in the second spatial plane. This is the temporal rate for three-dimensional motion in $\mathbf{D}_1$. From (2.17), (4.12), and (4.18), (E.1) can be transformed to a temporal distance in $\mathbf{D}_0$ thus

$$dx_0 = cu\left(\frac{d\tau_p}{d\tau}\right)dt = \left[c^2u^2\left(dt\right)^2 - \left(\frac{dr}{u}\right)^2 - (r+\alpha)^2\left\{(d\phi)^2 + (d\beta)^2\sin^2\phi\right\}\right]^{1/2} \tag{E.2}$$

where dx_0 is the distance moved along the temporal axis in an element of time dt in $\mathbf{D}_0$.
Incorporating (4.7), with (4.18) incorporated therein, converts (E2) to

$$\begin{aligned} dx_0 = \Bigg[c^2\left(\frac{r-\alpha}{r+\alpha}\right)(dt)^2 - \left(\frac{r+\alpha}{r-\alpha}\right)(dr)^2 \\ - (r+\alpha)^2\left\{(d\phi)^2 + (d\beta)^2\sin^2\phi\right\}\Bigg]^{1/2} \end{aligned} \tag{E.3}$$

as derived in [2], pp194, Eq(57.64) for the metric of the space-time of the General Relativity in the co-ordinate axes of Pseudo-Euclidean Space-Time. The above process shows that the metric of the General Theory is directly proportional to the temporal rate of a gravitating mass in $\mathbf{D}_1$. This suggests that the metric of the General Theory is a temporal metric rather than one of a true space-time interval.
Nevertheless, however (E.3) is interpreted, from the above it clearly involves three-dimensional spatial terms and, as such, can only represent the metric of the General Theory for the case in which three-dimensional spatial variation in position is involved. If this variation is put to zero, then a metric for a spatially stationary point in the co-ordinate system of the General Theory is obtained. Thus by putting dr, $d\phi$ and $d\beta$ to zero in (E.3) gives

$$dx_0 = c\left(\frac{r-\alpha}{r+\alpha}\right)^{1/2}dt \tag{E.4}$$

Re-inserting (4.7) and (4.18) then gives

$$dx_0 = cudt \tag{E.5}$$

so that the proper time of this point relative to Pseudo-Euclidean Space-Time is then

$$d\tau = \frac{dx_0}{c} = udt \tag{E.6}$$

as derived in (2.18).
Thus the proper time of a spatially stationary point in the space-time of the General Theory, relative to Pseudo-Euclidean Space-Time, is identical to the proper time of $\mathbf{D}_1$ relative to $\mathbf{D}_0$.

APPENDIX F

Radial and Radial-Normal Unit Vector Differentials in $\mathbf{D}_1$.

In this Appendix, proofs of the differentials of unit vectors in $\mathbf{D}_1$ as represented by (2.7) and (2.8) are given.
Consider the vector $\bar{\sigma}$ in $\mathbf{D}_1$.

$$\bar{\sigma} = \sigma\ \boldsymbol{n} \tag{F.1}$$

Differentiating this with respect to the time in $\mathbf{D}_1$.

$$\begin{aligned}\frac{d\bar{\sigma}}{d\tau} &= \frac{d\sigma}{d\tau}\boldsymbol{n} + \sigma\frac{d\boldsymbol{n}}{d\tau}\\ &= \dot{\sigma}\boldsymbol{n} + \omega\sigma\frac{d\boldsymbol{n}}{d\phi}\end{aligned} \tag{F.2}$$

Assume now that there is only radial normal motion, i.e. $\dot{\sigma} = 0$, then

$$\frac{d\bar{\sigma}}{d\tau} = \omega\sigma\frac{d\boldsymbol{n}}{d\phi} \tag{F.3}$$

Because this motion is only a radial normal one, the right hand side can be equated to a simple velocity term thus

$$\omega\sigma\frac{d\boldsymbol{n}}{d\phi} = \upsilon\boldsymbol{t} \tag{F.4}$$

This must be valid for all values of ω including boundary conditions. The lower condition is trivial, (when $\omega = 0, \upsilon = 0$), but at the upper condition of Terminal Spatial Velocity in the radial normal direction, i.e. $\omega\sigma = c$, the left hand side of (F.4) becomes

$$\left[\omega\sigma\frac{d\boldsymbol{n}}{d\phi}\right]_{upper} = c\frac{d\boldsymbol{n}}{d\phi} \tag{F.5}$$

At this boundary, temporal velocity is zero and spatial velocity is equal to the magnitude of Existence Velocity and therefore the right hand side of (F.4) can be written

$$[\upsilon\boldsymbol{t}]_{upper} = cu\boldsymbol{t} \tag{F.6}$$

Thus from (F.5) and (F.6)

$$\frac{d\boldsymbol{n}}{d\phi} = u\boldsymbol{t} \tag{F.7}$$

A similar proof exists for

$$\frac{d\boldsymbol{t}}{d\phi} = -u\boldsymbol{n} \tag{F.8}$$

These relationships exist because the Spatial Terminal Velocity in the radial normal direction is different from the magnitude of Existence Velocity in this Domain.

APPENDIX G

Reduction of Selected Relativistic Gravitational Equations to their Equivalents in Classical Theory.

This is only effected for the more complex expressions, or in trivial cases, where a special implication is involved. First it should be noted from (4.7) and (4.18) that the gravitational radius of a gravitational source can be expressed as

$$\alpha = r\frac{(1-u^2)}{(1+u^2)} \tag{G.1}$$

so that when $u = 1$, $\alpha = 0$ and therefore, from (4.18) and (4.21)

$$\sigma = r \quad and \quad \tau = t, \quad so\ that \quad \dot{\sigma} = \dot{r} \tag{G.2}$$

Section 2.

(i) Eq.(2.11), Existence Velocity

(a) Reduction to the Special Relativistic version is effected by putting $u = 1$

$$\mathbf{V} = \dot{r}\boldsymbol{n} + \omega r\boldsymbol{t} + \boldsymbol{j}c\left(1 - \frac{\dot{r}}{c^2} - \frac{\omega r}{c^2}\right)^{1/2} \tag{G.3}$$

(b) Reduction to the classical equivalent. In (G3) when $c \to \infty$

$$\mathbf{V} = \dot{r}\boldsymbol{n} + \omega r\boldsymbol{t} + \boldsymbol{j}\infty \tag{G.4}$$

as found in [1] and in classical studies the temporal term is ignored.

Section 3.

(ii) Eq.(3.6), Mass

(a) Reduction to the Special Relativistic version is effected by putting $u = u_0 = 1$

$$m = m_0\frac{\left(1 - \frac{\omega_0^2 r_0^2}{c^2}\right)^{1/2}}{\left(1 - \frac{\dot{r}^2}{c^2} - \frac{\omega^2 r^2}{c^2}\right)^{1/2}} \tag{G.5}$$

This can be compared with [1], Eq.(3.7) by putting $\omega = \omega_0 = 0$.

(b) Reduction to the classical equivalent. When in (G.5) $c \to \infty$

$$m = m_0 \tag{G.6}$$

(iii) Eq.(3.10), Rate of change of momentum.

(a) Reduction to the Special Relativistic version is effected by putting $u = 1$

$$\frac{d\mathbf{M}}{d\tau} = m\frac{\left(\ddot{r} - \omega^2 r\right)\boldsymbol{n}}{\left(1 - \frac{\dot{r}^2}{c^2}\right)} \tag{G.7}$$

which becomes with insertion of (G.5)

$$\frac{d\mathbf{M}}{d\tau} = m_0\frac{\left(\ddot{r} - \omega^2 r\right)\boldsymbol{n}}{\left(1 - \frac{\dot{r}^2}{c^2}\right)\left(1 - \frac{\dot{r}^2}{c^2} - \frac{\omega^2 r^2}{c^2}\right)^{1/2}} \tag{G.8}$$

This can be compared with [1], Eq.(3.9) by putting $\omega = 0$.
(b) Reduction to the classical equivalent. When in (G.8) $c \to \infty$

$$\frac{d\mathbf{M}}{dt} = m_0\left(\ddot{r} - \omega^2 r\right)\boldsymbol{n} \tag{G.9}$$

(iv) Eq.(3.14), Spatial gradient of energy.
(a) Reduction to the Special Relativistic equivalent is effected by putting $u = 1$

$$\frac{dE}{dr} = m\frac{\left(\ddot{r} - \omega^2 r\right)}{\left(1 - \frac{\dot{r}^2}{c^2}\right)} \tag{G.10}$$

which is the same as the magnitude of (G.7) and therefore shows that gravitation only exists within the Special Theory of Relativity as an axiomatic addition as it does in classical theory.

Section 5.
The planetary orbit. This is most easily reduced to the classical equivalent by first putting $\alpha = 0$ in (5.30) which gives

$$\phi = \chi \tag{G.11}$$

So that this gives in (5.28)

$$\Omega = 0 \tag{G.12}$$

and therefore in (5.11)

$$\frac{1}{r} = \frac{1}{L}\left(1 + e\cos\phi\right) \tag{G.13}$$

the equation of a standard conic section, and in which the eccentricity, e, is reduced from (5.24) to

$$e = \frac{m_0 h^2 \mu_0}{F_0} - 1 \tag{G.14}$$

where

$$\frac{F_0}{m_0\mu_0} = \gamma m_g \mu_0 \tag{G.15}$$

and where now $\mu_0 = 1/r_0$
(v) Eq.(5.8), equation of the orbit.
First express (5.8) as

$$\frac{d^2\mu}{d\phi^2} + \mu = \frac{\gamma m_g}{h^2} + 3\alpha\mu^2 \tag{G.16}$$

To reduce (G.16) to its classical equivalent put $\alpha = 0$ and then put

$$\gamma m_g = \frac{F}{m_0\mu^2} \tag{G.17}$$

to yield

$$\frac{d^2\mu}{d\phi^2} + \mu = \frac{F}{m_0 h^2\mu^2} \tag{G.18}$$

the classical equation in mechanics.

Appendix A
(vi) Eq.(A.4), The equation of free planar motion in the axes of D_0.

Substituting for α from (4.6) gives

$$\frac{d^2r}{dt^2} = -\frac{\gamma m_g\left(r - \frac{\gamma m_g}{c^2}\right)}{\left(r + \frac{\gamma m_g}{c^2}\right)^3} + \frac{3\gamma m_g\left(\frac{dr}{dt}\right)^2}{c^2\left(r^2 - \frac{\gamma^2 m_g^2}{c^4}\right)} + \left(r - \frac{\gamma m_g}{c^2}\right)\left(\frac{d\phi}{dt}\right)^2 \tag{G.19}$$

and then assuming c to be infinitely large reduces this to the classical equation

$$\frac{d^2r}{dt^2} = -\frac{\gamma m_g}{r^2} + r\left(\frac{d\phi}{dt}\right)^2 \tag{G.20}$$

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